CHAPTER

Connectedness

Intuitively a connected space is a topological space which consists of a single piece. It is the simplest property of a topological but it is one of the most important properties for the application of topology to analysis and geometry

Significance of Connectedness

Connected is significance because it has many application for analysis and geometry e.g

1): Intervals are connected subspaces on the real line. we shall see that they are the only connected subspaces

2): Continuous functions are often defined on the intervals and such functions have many pleasant properties e.g such a function assumes as a value every number between any two of its values (Intermediate Value Theorem) further graph of a such function is a connected subspace of the Euclidean plane

3) Connectedness is also a basic notion in Complex analysis. for regions on which analytic functions are studied are generally taken to be connected open subspaces of the complex plane

4) In the portion of topology which deals with curves and their properties, connectedness is of great significance because a continuous curve is certainly a connected topological space.

5): disconnected spaces are also interesting e.g The Cantor set is disconnected to the extreme degree.

Separated Sets

Two Subsets A & B of a top-space are separated if
$A \cap \bar{B} = \phi$, $\bar{A} \cap B = \phi$
or Equivalently $(A \cap \bar{B}) \cup (\bar{A} \cap B) = \phi$
(Hausdorff - Lennaes Condition)

Remarks # Separated set are of course disjoint, but disjoint sets need not be separated e.g sets $A = [0\ 1]$ and ~~B=(1~~ $B = (1, 2)$ are not separated since 1 is a limit point of B. However the sets $A = (0, 1)$ & $B = (1, 2)$ are separated.

Connected Topological Space

A top-space X is called connected if it cannot be expressed as union of non-empty separated sets i.e if there do not exist non-empty separated sets A & B such that

$$X = A \cup B$$

Disconnected Topological Space

A top-space X is disconnected if it is not connected i.e if it can be expressed as union two non-empty separated sets.

OR

A top-space X is disconnected if there exists two non-empty separated sets A & B such that

$$X = A \cup B.$$

Thus sets A & B form a partition or separation or (dischim) disconnection and is denoted by {A, B} and some times by A | B

Theorem# A top space (X, T) is connected iff X cannot be expressed as union of two disjoint open sets.

Proof# Let (X, T) be connected. We are to prove that it cannot be expressed as union of two disjoint open sets.

Let on the contrary ∃ open sets U & V such that

$X = U \cup V$ & $U \cap V = \phi$ (disjoint)

Then clearly no point of V is a limit point of U because if V contains some limit point of U, then V being an open set containg that point must intersect with U in other than that point. But $U \cap V = \phi$

Hence

$$\bar{U} \cap V = \phi \quad \& \quad U \cap \bar{V} = \phi$$

Thus X is a union two separated sets U & V and so is disconnected. This is a contradiction. Hence X can not be expressed as union two disjoint open sets.

Conversely let X cannot be expressed as disjoint union of two open sets. We are to proe that X is connected. Let on the contrary X is not connected. Then ∃ disjoint sets A & B such that

$X = A \cup B$ & $\bar{A} \cap B = \bar{B} \cap A = \phi$ separated sets

Now $\bar{A} \cap B = \phi$ implies

$(\bar{A} \cap B)^c = \phi^c$

$(\bar{A})^c \cup B^c = X$

$\Rightarrow B^c = X - (\bar{A})^c = \bar{A}$ $\quad \Rightarrow B = (\bar{A})^c = X - \bar{A}$

Similarly $A = X - \bar{B}$

$\Rightarrow$ A & B are both open sets

Thus X is represented as union of two disjoint open sets which is a contradiction. Hence X is connected.

Remarks # from this theorem we note that an equivalent definition of connected space and disconnected space can be given as under.

Connected Top-Space

A top-space X is said to be connected if it can not be expressed as disjoint union of two open sets i.e if there do not exist open set G & H such that.

$$X = G \cup H \quad \& \quad G \cap H = \phi$$

{G, H} is called a disconnection of X

Disconnected Top-Space

A top-space X is disconnected if it can be expressed as disjoint union of two open sets i.e if there exist two open sets G & H such that

$$X = G \cup H \quad \& \quad G \cap H = \phi$$

{G, H} is called a disconnection of X

Remarks If a top-space (X, T) has a disconnection {G, H}. Such G, H are open and disjoint i.e

$$X = G \cup H \quad \& \quad G \cap H = \phi$$

$\Rightarrow G^c = H$ & $H^c = G$

$\Rightarrow$ G & H are complements of each other

Therefore both are open as well as closed

Margin notes:

$\bar{A}, \bar{B}$ closed and A, B are open.



$\because G \cap H = \phi$

$G^c = H$, $H^c = G$ (open closed)

$X = G^c \cup H^c$

$\phi = G^c \cap H^c$

disjoint

$G \cap H = \phi$

$G^c \cap H^c = X$

Further since G & H are open G^c & H^c are closed

$\because\ G \neq \phi \qquad H \neq \phi \qquad G \neq X,\ H \neq X$

$\therefore\ G' \neq \phi \qquad H' \neq \phi$

Now

$$X = G \cup H \qquad \& \ G \cap H = \phi$$

$$\Rightarrow G' \cap H' = \phi \qquad \Rightarrow G' \cup H' = X$$

Thus if X is disconnected, then it is disjoint union of two closed sets and conversely.

So

X is disconneted $\Leftrightarrow$ it can be represeted as union of two non-empty open (closed) disjoint subsets of X.

discrete $\tau_1(X, \phi)$ Connected

Theorem# A top-space is connected iff the only subsets of X that are both open and closed are X and ϕ OR A top-space is connected iff it has no non-empty proper subsets which are both open and closed.

Proof# Let X be connected. Then X cannot be expressed as union two non-empty disjoint proper subsets of X. اگر proper set بنتے تو وہ بیک وقت open اور closed ہوتے۔

$\Rightarrow$ There does not exist non-empty disjoint proper open subsets of X whose union is X. If such open sets would exist, they would be both open and closed.

Hence X & ϕ are only sets which are both open and closed.

Conversely let ϕ & X be the only sets which are both open and closed. We are to prove that X is connected.

Let X be disconnected and the open sets (proper) A & B make a partition for X i.e (non-empty & disjoint)

$$X = A \cup B \qquad \& \ A \cap B = \phi$$

closed

$\Rightarrow A^c = B \ ,\ B^c = A$ — open

$\Rightarrow A$ & B are both open and closed. This is a

Contradiction to hypothesis that X & ϕ are only sets which are both open and closed. Hence X is connected.

Examples # 1

(i) Let $X = \{a, b, c\}$

$\mathcal{T} = \{\phi, \{a\}, \{b, c\}, X\}$

put $G = \{a\}$ $H = \{b, c\}$ — disjoint

Then $X = G \cup H = \{a, b, c\}$

$G \cap H = \{a\} \cap \{b, c\} = \phi$

$G \neq \phi$ $H \neq \phi$ $G \in \mathcal{T}$ $H \in \mathcal{T}$

Hence $\{G, H\}$ is a disconnection for $(X, \mathcal{T})$

Example # 2

Let $X = \{a, b, c\}$

$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, X\}$

Empty set & singleton set are connected.

- The non-empty open sets $\{a\}$ & $\{a, b\}$ are such that

$\{a\} \cap \{a, b\} = \{a\}$ $X \neq \{a\} \cup \{a, b\}$

So $(X, \mathcal{T})$ has no

Example 3 # The empty set and any singleton set are connected

Since there are no non-empty sets A & B which are separated and such that

$\phi = A \cup B$

$\Rightarrow$ ϕ is connected.

OR

There are no two non-empty open (closed) sets whose union is ϕ, therefore ϕ is connected.

(ii) Let $\{p\}$ be any singleton subset of Top-space. Then

$\{p\} \neq A \cup B$ $A \neq \phi$ $B \neq \phi$

$A \cap B = \phi$ and A & B are both open or both closed. Hence $\{p\}$ is a connected set

separate 5 singleton ... at least double

disjoint

Note $A \cup B$ can never be a singleton, it is at least a double-ton

Example # Every infinite set with co-finite topology is connected.

Sol # Let X be an infinite set with co-finite topology. To prove that X is connected. Suppose on the contrary that X is disconnected. Then there are two disjoint, non-empty open sets A & B s.t

$$X = A \cup B$$

Since A & B are both open, they belong to co-finite topology, i.e A' & B' are finite.

Also

$$\because \quad A \cap B = \phi$$

$$\Rightarrow \quad X = A' \cup B' \quad (A' \,\&\, B' \text{ both finite})$$

$\Rightarrow$ X being union of finite sets is finite which is a contradiction to given hypothesis that X is an infinite set. Hence X is connected.

Example #

Every indiscrete space is connected because in it ϕ, X are only ~~open~~ sets which are both open and closed.

Example Let $X = \{a, b, c\}$

$$\mathcal{T} = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b\}, \{c\}\}$$

Then

$$\{a, c\} \cup \{b\} = X \quad \& \quad \{a, c\} \cap \{b\} = \phi$$

actually any discrete space with more than one point is disconnected. In particular, any discrete space consisting of two points is disconnected.

Example # The set $\mathbb{Q}$ of rational numbers as a subspace of the real line is not Connected.
Here for any irrational number r
$$\{ \mathbb{R} \cap (-\infty, r), \mathbb{R} \cap (r, \infty) \}$$
is a disconnection is a disconnection for $\mathbb{Q}$.

Disconnection of Subset (subspace)

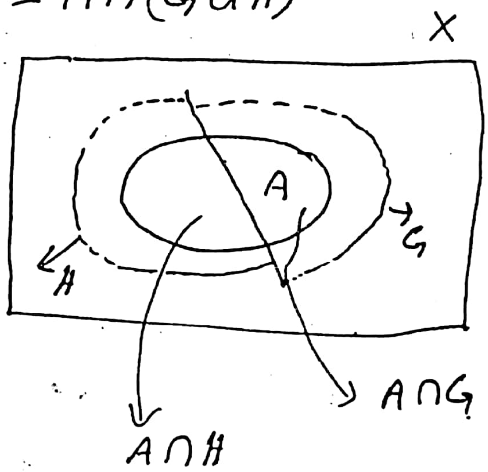
Let X be a topological space and $A \subseteq X$. A is said to be disconnected iff there exist two non-empty disjoint open subsets ~~of X~~ G & H of X such that

G, H open both Closed i.e. ...

$$A \subseteq G \cup H \qquad A \cap G \neq \phi \qquad A \cap H \neq \phi$$
$$A = (A \cap G) \cup (A \cap H) = A \cap (G \cup H)$$

and $(A \cap G) \cap (A \cap H) = \phi$ $\Rightarrow A \cap (G \cap H) = \phi$

This amounts to saying that $\{A \cap G, A \cap H\}$ form a disconnection for A w.r.t relative topology T_A i.e A is disconnected subset of (X, T) iff (A, T_A) is disconnected subspace.



× [Disconnected subset] ×

Connected Subset # A subset A of a top-space X is connected if it is not disconnected in the sense given above i.e if there donot exist two non-empty open subsets G, H of X such that
$$A \subseteq G \cup H \qquad A \cap G \neq \phi \qquad A \cap H \neq \phi$$
$$A = (A \cap G) \cup (A \cap H) = A \cap (G \cup H)$$

Equivalently a subset D of a top-space is disconnected iff there exist two non-empty sets A & B such that
$$A \cap \bar{B} = \phi \qquad \bar{A} \cap B = \phi \qquad D = A \cup B$$

i.e A is disconnected subset of (X,T) if (A, T_A) is disconnected subspace of (X,T)

Equivatently A subset D of (X,T) is disconnected iff there exist two non-empty sets A & B such that

$$A\cap\bar{B}=\phi \quad \bar{A}\cap B=\phi \quad D=A\cup B$$

obviously D is connected if the above condition is not satisfied.

Example # Let $X=\{a,b,c\}$
$\mathcal{T}=\{\phi, X, \{a\}, \{b,c\}\}$

The subset $A=\{a,c\}$ is disconnected because if $G=\{a\}$ $H=\{b,c\}$, then G, H are both are non-empty, disjoint, open and

$$A\subseteq G\cup H$$

Further $A\cap G=\{a\}$ $A\cap H=\{c\}$

$$(A\cap G)\cap(A\cap H)=\phi$$

$$A=(A\cap G)\cup(A\cap H)$$

on the other hand, the subset $B=\{b,c\}$ is connected because we can not split it into non-empty disjoint open subsets in the relative topology

Example # Consider the subset $A\subseteq R$ defined as

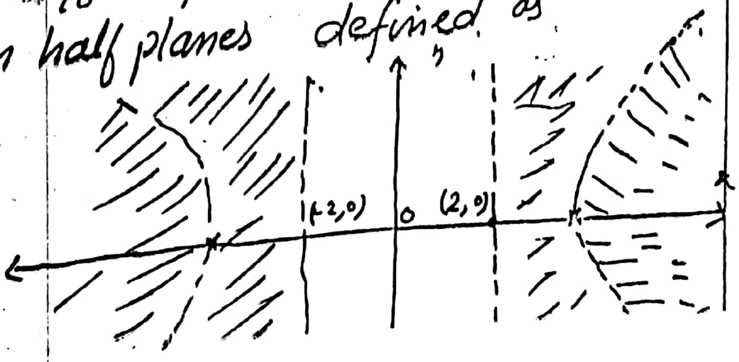
$$A=\{(x,y):\frac{x^2}{16}-\frac{y^2}{9}\geq 1\}$$

If G & H are open half planes defined as

$G=\{(x,y):x>2\}$

$H=\{(x,y):x<-2\}$

Then G, H form a disconnection



Note # We note that $A = (A \cap G) \cup (A \cap H)$

$\Rightarrow A = A \cap (G \cup H) \subseteq G \cup H$

$\phi = (A \cap G) \cap (A \cap H) \Leftrightarrow G \cap H \subseteq A^c$

Thus $G \cup H$ is a disconnection of A iff

$A \cap G \neq \phi$ $A \cap H = \phi$ $A \subseteq G \cup H$

, $G \cap H \subseteq A^c$

→ # Prove that connectedness is absolute property.

Theorem # If E is a subset of a subspace (Y, T_Y) of a topological space (X, T), then A T_Y-Connected iff it is T-Connected.

Proof # Let A & B be two non-emty, disjoint sets whose union is E. Then $E \subseteq (Y, T_Y) \subseteq (X, T)$

$A, B \subseteq Y \subseteq X$

Now $E = A \cup B$ and $[A \cap \bar{B}] \cup [\bar{A} \cap B] = \phi$

from Hausdorff-Lennes Separation Condition we find that

$[A \cap \bar{B}] \cup [\bar{A} \cap B] = [A \cap Y \cap \bar{B}] \cup [\bar{A} \cap Y \cap B]$

$= [A \cap \bar{B}_Y] \cup [\bar{A}_Y \cap B]$

Thus if the Condition is satisfied with respect to one topology, it is satisfied with respect to the other

OR

In order to prove that E have a partition w.r.t either space X or Y, E must be union two non-empty disjoint subsets. Let A & B be non-empty disjoint subsets of Y. Then $A, B \subseteq Y \subseteq X$.

Let E be connected relative to Y. We are to prove that E is connected relative to X. Let on the contrary, E be disconnected in X. Then there exist two non-empty separated set A & B such that

Then $E = A \cup B$ such that

Now A & B are separated in X. Therefore

$[A \cap (\bar{B})_X] \cup [(\bar{A})_X \cap B] = \phi$

$[A \cap Y \cap (\bar{B})_X] \cup [(\bar{A})_X \cap Y \cap B]$ $\therefore Y \cap B$

$$\{A \cap (\bar{B})_X\} \cup \{(\bar{A})_X \cap B\} = \phi$$

$$\{A \cap Y \cap (\bar{B})_X\} \cup \{(\bar{A})_X \cap B \cap Y\} = \phi$$

$[\because A \cap Y = A$, $B \cap Y = B]$

$$[A \cap (\bar{B})_Y] \cup [(\bar{A})_Y \cap B] = \phi \qquad [Y \cap \bar{B}_X = (\bar{B})_Y$$

$\Rightarrow E$ is disconnected in Y, which is a contradiction. Thus if E connected in Y, then it is connected in X.

<u>Converse</u> # Let E be connected in X and E be disconnected in X.

Then $E = A \cup B$ such that

$$\{A \cap (\bar{B})_Y\} \cup \{(\bar{A})_Y \cap B\} = \phi$$

or

$$[A \cap Y \cap (\bar{B})_X] \cup \{(\bar{A})_X \cap Y \cap B\} = \phi$$

$$[A \cap (\bar{B})_X] \cup [(\bar{A})_X \cap B] = \phi$$

$\Rightarrow$ E is disconnected in X which is a contradiction

Hence when E is connected in X, then it is connected in Y

same

<u>Comments</u> # from this Theorem we note that connectedness is an <u>absolute property</u> of sets i.e it <u>does not depend</u> on the space in which the set is contained except that the topology must, of course, be the <u>relative topology</u>. Because of this usually connectedness of the entire space is considered.

Theorem # Let A be connected subset of a top-space (X, τ). If there is a disconnection $G \cup H$ for X, then either $A \subseteq G$ or $A \subseteq H$ i.e either $A \cap G = \phi$ or $A \cap H = \phi$

connected $A \subseteq X$ — disconn $X = G \cup H$. A is either contained in G or H.

Proof # Suppose that $A \cap G \neq \phi$ $\quad A \cap H \neq \phi$

$\because$ $G \cup H$ is a disconnection for X

$\therefore$ $X = G \cup H$

$\Rightarrow A \subseteq G \cup H$

$(A \cap G) \cup (A \cap H) = A \cap (G \cup H)$ (disconn., then sets)

$= A \cap X$

$= A$

Thus

$A = (A \cap G) \cup (A \cap H)$ $\quad A \cap G \neq \phi$, $A \cap H \neq \phi$

and $A \subseteq G \cup H$

By definition A is disconnected which is a contradiction because A is connected.

Therefore $A \cap G = \phi$ or $A \cap H = \phi$

$\Rightarrow A \subseteq H$ or $A \subseteq G$

Theorem # Let $G \cup H$ be a disconnection of A. Let $B \subseteq A$ such that B is connected. Then either $B \cap H = \phi$ or $B \cap G = \phi$ and so either $B \subseteq G$ or $B \subseteq H$.

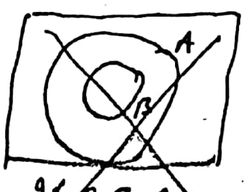
Proof # $\because$ $B \subseteq A$ & $A \subseteq G \cup H$

$\therefore$ $B \subseteq G \cup H$

$B \subseteq A$ & $A \subseteq G \cup H$

$\Rightarrow A^c \subseteq B^c$ & (crossed out: $(G \cup H)^c \subseteq A^c$, $G^c \cap H^c \subseteq A^c$, $G \cap H \subseteq A^c$)

(X.T)



If $B \subseteq A$
The $A^c \subseteq B^c$

Also $\Rightarrow G \cap H \subseteq A^c \subseteq B^c$

Now $\Rightarrow B \subseteq G \cup H$

Let $B \cap H \neq \phi$ $\quad B \cap G \neq \phi$

$\Rightarrow G \cup H$ is a disconnection of B. But B is connected.

Hence either $B \cap H = \phi$ or $B \cap G = \phi$

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Theorem # Let (X, T) be a top-space. If C is a connected subset of a disconnected set D with disconnection $G \cup H$, prove that $C \subseteq G$ or $C \subseteq H$

Proof # $G \cup H$ forms a disconnection of D means that G & H are non-empty open subsets of X such that

$D_1 = D \cap G$ & $D_2 = D \cap H$ are non-empty disjoint sets with

$D = D_1 \uplus D_2$, $D \subseteq G \cup H$

And $D \cap G \cap H = \phi$ or $G \cap H \subseteq D'$

Now $C \subseteq D$ & $D \subseteq G \cup H$

$\Rightarrow C \subseteq G \cup H$

and $C \subseteq D$ & $G \cap H \subseteq D'$ — $C \subseteq D \Rightarrow D' \subseteq C'$

$\Rightarrow G \cap H \subseteq C'$

It follows that if $C \notin G$ and $C \notin H$ i.e

if $C \cap H \neq \phi$ and $C \cap G \neq \phi$

, then $G \cup H$ will form a disconnection for the set C, which is a connected subset of X. This being untrue, we must have $C \cap G = \phi$ or $C \cap H = \phi$ i.e

either $C \subseteq H$ or $C \subseteq G$.

Now we prove that $C \cap G = \phi \Rightarrow C \subseteq H$ &

$C \cap H = \phi \Rightarrow C \subseteq G$.

(i) Let $C \cap G = \phi$. Then

$C \subseteq G \cup H \Rightarrow C = C \cap (G \cup H)$

$= \{C \cap G\} \cup \{C \cap H\}$

$= \phi \cup \{C \cap H\} = C \cap H$

$[L \subseteq M \Rightarrow L = L \cap M$

Thus $C = C \cap H \Rightarrow C \subseteq H$

(ii) Let $C \cap H = \phi$, then

$C \subseteq G \cup H \Rightarrow C = C \cap (G \cup H) = \{C \cap G\} \cup \{C \cap H\}$

$= \{C \cap G\} \cup \phi = C \cap G$

Thus $C = C \cap G \Rightarrow C \subseteq G$

Consequently either $C \subseteq G$ or $C \subseteq H$

Theorem # Let A & B be two connected subsets of a top-space (X, T) such that $A \cap B \neq \phi$, Then $A \cup B$ is also connected.

Proof # Suppose that $E = A \cup B$ and E is disconnected. Then $\exists$ two non-empty disjoint open sets G & H such that

$$\left.\begin{array}{l} E \subseteq G \cup H, \quad E \cap G \neq \phi \quad E \cap H \neq \phi \\ \text{and} \quad E \cap G \cap H = \phi \end{array}\right\} \rightarrow (1)$$

$\because$ $A \subseteq E \subseteq G \cup H$ and it is given that A is connected

$\therefore$ $A \cap G = \phi$ or $A \cap H = \phi$ (By above Theorem) $\rightarrow (2)$

$\Rightarrow A \subseteq H$ or $A \subseteq G$

Similarly $B \subseteq E \subseteq G \cup H$ implies that either $B \cap G = \phi$ or $B \cap H = \phi$

$\Rightarrow B \subseteq H$ or $B \subseteq G$ $\rightarrow (3)$

$\Rightarrow$ Both A & B lies inside G or inside H or one in H & one in G

(i) Consider

$A \subseteq G$; $B \subseteq G$

Then $A \cup B \subseteq G$ & $(A \cup B) \cap H = \phi$ ($\because$ A, B are both in G or both in H & we have considered in G)

$\Rightarrow E \cap H = \phi$

which is a contradiction to $E \cap H \neq \phi$

(ii) Let $A \subseteq H$, $B \subseteq H$

Then $A \cup B \subseteq H$ & $(A \cup B) \cap G = \phi$

$\Rightarrow E \cap G = \phi$

which is a contradiction. Hence E is connected.

iii) Let $A \subseteq G$ and $B \subseteq H$

Then $A \cap H = \phi$ & $B \cap G = \phi$

$E \cap G = (A \cup B) \cap G = \{A \cap G\} \cup \{B \cap G\} = \{A \cap G\} \cup \phi$

$= A \cap G = A$ ($\because A \subseteq G$)

$E \cap H = (A \cup B) \cap H = \{A \cap H\} \cup \{B \cap H\} = B \cap H = B$

In this case

$$A \cap B = (E \cap G) \cap (E \cap H)$$
$$= E \cap (G \cap H) = E \cap \phi = \phi$$

which is a Contradiction because $A \cap B \neq \phi$

(iv) Let $A \subseteq H$ & $B \subseteq G$

Then $A \cap G = \phi$. $B \cap H = \phi$

$$E \cap G = (A \cup B) \cap G = (A \cap G) \cup (B \cap G)$$
$$= \phi \cup (B \cap G) = B \cap G = B$$

$$E \cap H = (A \cup B) \cap H = (A \cap H) \cup (B \cap H)$$
$$= (A \cap H) \cup \phi = A \cap H = A$$

Again $A \cap B = \phi$

which is a contradiction to $A \cap B \neq \phi$

Consequently our supposition that $E = A \cup B$ is disconnected is wrong and so E is Connected

Note cases (iii) & (iv) are unnecessary

Remarks # The above Theorem may be generalized as in the theorem below

Theorem # If $\{E_\alpha : \alpha \in I\}$ is a family of connected subsets of a top-space X such that $\bigcap_{\alpha \in I} E_\alpha \neq \phi$, Then $\bigcup_{\alpha \in I} E_\alpha$ is also connected.

Proof # Let $E = \bigcup_{\alpha \in I} E_\alpha$ and E is disconnected.

Let $\{A, B\}$ be a disconnection of X i.e

$E = A \cup B$ $\quad A \cap B = \phi$, $A \neq \phi$, $B \neq \phi$

$\because$ $E_\alpha \subseteq E$ $\forall \alpha$ and E_α is connected for all α

$\therefore$ each E_α is either in A or in B (By previous Theorem)

Since $\bigcap_{\alpha \in I} E_\alpha \neq \phi$, all E_α are contained in A or in B because if for some $\alpha \in I$ $E_\alpha \subseteq A$, for $\beta \in I$ $E_\beta \subseteq B$

Then $E_\alpha \cap E_\beta = \phi$ ($\because A \cap B = \phi$) which is not possible because all E_α have at least one point Common.

$\Rightarrow E \subseteq A$ or $E \subseteq B$

Let $E \subseteq A$, then $B = \phi$ $\quad (\because E = A \cup B)$

If $E \subseteq B$, then $A = \phi$

a Contradiction. Hence E is connected.

Theorem # If A & B are two connected subsets of a top-space and are not separated, then $A \cup B$ is connected.

Proof # Let $E = A \cup B$ be disconnected and $G \cup H$ be its disconnection.

Then A is connected subset of disconnected set E.

Therefore

$$A \subseteq G \quad \text{or} \quad A \subseteq H.$$

Similarly B is a connected subset of E and so

$$B \subseteq G \quad \text{or} \quad B \subseteq H$$

Now A & B may or may not be disjoint

If $A \cap B \neq \phi$, then both A & B are either in G or in H.

Case I Let $A \subseteq G$, $B \subseteq G$

Then $A \cup B \subseteq G \Rightarrow E \subseteq G$, $\Rightarrow E \cap H = \phi$

which is a contradiction to $E \cap H \neq \phi$

Case II Let $A \subseteq H$, $B \subseteq H$

Then $A \cup B \subseteq H \Rightarrow E \subseteq H \Rightarrow E \cap G = \phi$

which is a contradiction

If $A \cap B = \phi$, then one of A, B is in G and other is in H

Case III Let $A \subseteq G$, $B \subseteq H$

$\Rightarrow A \cap H = \phi$ & $B \cap G = \phi$

$E \cap G = (A \cup B) \cap G = \{A \cap G\} \cup \{B \cap G\}$

$= \{A \cap G\} \cup \phi = A \cap G = A \quad (\because A \subseteq G$

$E \cap H = (A \cup B) \cap H = (A \cap H) \cup (B \cap H)$

$= \phi \cup (B \cap H) = B \cap H = B$

Case IV Let $A \subseteq H$, $B \subseteq G$

Then $A \cap G = \phi \quad B \cap H = \phi$

13+4-1;

$E \cap G = (A \cup B) \cap G = (A \cap G) \cup (B \cap G)$.
$= \phi \cup (B \cup G)$
$= B \cup G = B$

$E \cap H = (A \cup B) \cap H = (A \cap H) \cup (B \cap H) = A \cap H = A$

Since $G \cup H$ is a disconnection of E, it follows that $E \cap G$, $E \cap H$ are separated sets.
Consequently A & B are also separated. This contradicts the hypothesis. Hence E is not disconnected but is connected.

Theorem# If A & B are non-empty separated subsets of a top-space X, $A \cup B$ is disconnected.

Proof# Since A & B are separated, therefore

$A \cap \bar{B} = \phi \qquad \bar{A} \cap B = \phi$

Let $G = \bar{B}^c \qquad H = \bar{A}^c$

Then G & H are open and

$(A \cup B) \cap G = (A \cap G) \cup (B \cap G)$
$= (A \cap \bar{B}^c) \cup (B \cap \bar{B}^c)$
$= A \cup \phi = A$

$(A \cup B) \cap H = (A \cap H) \cup (B \cap H)$
$= (A \cap \bar{A}^c) \cup (B \cap \bar{A}^c)$
$= \phi \cup B = B$

$A \cap \bar{B} = \phi$
$\Rightarrow A \subseteq \bar{B}^c$
$\because B \subseteq \bar{B}$
$\therefore (\bar{B})^c \subseteq B^c$
$B \cap (\bar{B})^c \subseteq B \cap B^c = \phi$
$\Rightarrow B \cap (\bar{B})^c = \phi$

$\because$ A & B are separated
$\therefore A \cap B = \phi$

$\{(A \cup B) \cap G\} \cap \{(A \cup B) \cap H\} = A \cap B = \phi$

Also

$\{(A \cup B) \cap G\} \cup \{(A \cup B) \cap H\} = A \cup B$

Also since A & B are separated, $(A \cup B) \cap G$ & $(A \cup B) \cap H$ are also separated.
Hence $\{G, H\}$ form a disconnection of $A \cup B$ and so $A \cup B$ is disconnected.

Theorem# Let $G \cup H$ be a disconnection of A, then $A \cap G$, $A \cap H$ are separated sets OR A has a separation

$$A = (A \cap H) \cup (A \cap G) \quad (\text{union of separated sets})$$

Proof# Since $G \cup H$ is a disconnection of A, therefore

$$A = (A \cap G) \cup (A \cap H) \quad \& \quad (A \cap G) \cap (A \cap H) = \emptyset$$

and G, H are open subsets of X

Let $P = A \cap G$ $\quad Q = A \cap H$

We are to show that

$$P \cap \bar{Q} = \emptyset \quad \& \quad \bar{P} \cap Q = \emptyset$$

Let $x \in P^d$ and suppose that $x \in Q = A \cap H$. Then $x \in H$. Since x is a limit point of P and H is an open set containg x, H must contain a point $P = A \cap G$ other than x. This implies that

$$(A \cap G) \cap H \neq \emptyset$$

But $(A \cap G) \cap (A \cap H) = \emptyset$

$\Rightarrow A \cap (G \cap H) = \emptyset$

$\Rightarrow (A \cap G) \cap H = \emptyset$

a contradiction. Hence Q contains no limit point of P. Similarly we can prove that P contains no limit point of Q. Thus

$$\bar{P} \cap Q = \emptyset \qquad P \cap \bar{Q} = \emptyset$$

$\Rightarrow P = A \cap G$ $\quad Q = A \cap H$ are separated sets.

Theorem# Let (X, T) be a connected top-space and C be a connected subset of X. If $C^c = A \cup B$ (A & B are separated) i.e if $\{A, B\}$ is a separation of $X \setminus C = C^c$, then show that $A \cup C$ & $B \cup C$ are connected.

Proof Suppose that $A \cup C$ is not connected and has separation G & H.

$\because C \subseteq A \cup C$ $\quad \therefore$ a connected set C is a subset

of disconnected set $D = A \cup C$, we must have $C \subseteq G$ or $C \subseteq H$ (Sep. son rule)

Let $C \subseteq G$. Then

$C \subseteq G \Rightarrow C \cup A \subseteq G \cup A$

$\Rightarrow G \cup H \subseteq G \cup A$ [$C \cup A = G \cup H$]

$\Rightarrow H \subseteq A$

Further A & B are separated and

$H \subseteq A$ $B \subseteq B$

So H & B are separated. Thus H is separated from G as well as B and consequently, H is separated from $G \cup B$.

Now

$X \setminus C = C^c = A \cup B$

So $X = C \cup C^c = C \cup (A \cup B)$

$= (C \cup A) \cup B$

$= (G \cup H) \cup B$

$= (G \cup B) \cup H$

$\Rightarrow X$ is union two separated sets $G \cup B$ & H.

$\Rightarrow X$ is disconnected which is a contradiction.

It follows that $C \cup A$ must be connected.

Similar treatment shows that $C \cup B$ is connected.

Method - II

Suppose that $A \cup C$ is not connected and has a sepration G & H such that

$A \cup C = G \cup H$

$\because C \subseteq A \cup C$ and $A \cup C$ is disconnected with disconnection G & H

$\therefore C \subseteq G$ or $C \subseteq H$

Let $C \subseteq G$

Then $C \cup A \subseteq G \cup A$

$\Rightarrow G \cup H \subseteq G \cup A \Rightarrow H \subseteq A$

We prove that $\{G, H \cup B\}$ forms a separation of X

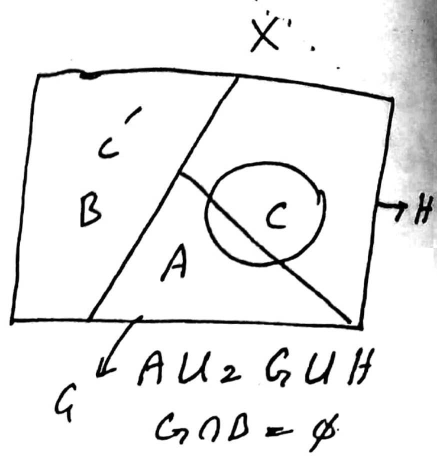
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leading to a contradiction.

$G \cap (H \cup B) = (G \cap H) \cup (G \cap B)$
$= \phi \cup \phi = \phi$

Also

$G \cup (H \cup B) = (G \cup H) \cup B$
$= A \cup C \cup B$
$= (A \cup B) \cup C$
$= C' \cup C = X$



and $G \neq \phi$ $H \cup B \neq \phi$

Now we prove that both are closed. Let x be a limit point of G.

∵ $\{G, H\}$ is a separation of $A \cup C$ | $A \cup C = G \cup H$

∴ x can not be in H | $C' = A \cup B$

Also x can not be in B because $\{A, B\}$ is a separation. Hence $x \in G$ | $x \in A \cup C$

and G becomes closed. | $x \notin B$

The limit points of $H \cup B$ are either limit point B or of H. No limit point of H is in G because $\{G, H\}$ is a separation. Also no limit point of B is in G, since $G \subseteq A$ and $\{A, B\}$ is a separation. Thus $\{G, H \cup B\}$ is a separation of X which is a contradiction. Hence $A \cup C$ is connected.

Similarly we can prove that $C \cup B$ is connected.

Theorem# Let (X, T) be a topological space and let A & B be subsets of X. If each of the sets A, B is closed in their union and if $A \cup B$ and $A \cap B$ are connected, then show that A is connected & B is connected.

Proof# If $A \subseteq B$ or $B \subseteq A$

Then $A \cup B = B$ or $A \cup B = A$

and hence A, B are connected because $A \cup B$ is connected.

Assume that $A \nsubseteq B$ and $B \nsubseteq A$

Then $A - B$ and $B - A$ are non-empty

$\because$ A & B are closed in $A \cup B$

$\therefore$ $(A \cup B) \setminus B = (A \setminus B) \cup (B \setminus B) = A \setminus B$ is open in $A \cup B$

Similarly $B \setminus A$ is open in $A \cup B$

Consequently $\{A \setminus B, B \setminus A\}$ forms a separation

Also

$$(A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A \cap B)^c \qquad \because A - B = A \cap B^c$$

$$= (A \cup B) \cap (A^c \cup B^c)$$
$$= \{(A \cup B) \cap A^c\} \cup \{(A \cup B) \cap B^c\}$$
$$= \{(A \cap A^c) \cup (B \cap A^c)\} \cup \{(A \cap B^c) \cup (B \cap B^c)\}$$
$$= \{\phi \cup (B \cap A^c)\} \cup \{(A \cap B^c) \cup \phi\}$$
$$= (B \cap A^c) \cup (A \cap B^c)$$
$$= (B \setminus A) \cup (A \setminus B)$$

Also

$$(A \setminus B) \cap (B \setminus A) = (A \cap B^c) \cap (B \cap A^c)$$
$$= A \cap A^c \cap B \cap B^c$$
$$= \phi \cap \phi = \phi$$

Thus $A \cap B$ is connected and $\{A \setminus B, B \setminus A\}$ is a separation of $(A \cup B) \setminus (A \cap B)$. So by above Theorem $(A \setminus B) \cup (A \cap B)$ & $(B \setminus A) \cup (A \cap B)$ are connected.

But

$$(A \setminus B) \cup (A \cap B) = (A \cap B^c) \cup (A \cap B)$$
$$= \{(A \cap B^c) \cup A\} \cap \{(A \cap B^c) \cup B\}$$
$$= \{(A \cup A) \cap (B^c \cup A)\} \cap \{(A \cup B) \cap (B \cup B^c)\}$$
$$= \{A \cap (B^c \cup A)\} \cap \{(A \cup B) \cap X\}$$
$$= \{(A \cap B^c) \cup (A \cap A)\} \cap (A \cup B)$$

$$(A \setminus B) \cup (A \cap B) = A$$

& $(B \setminus A) \cup (A \cap B) = B$

Hence A & B are connected.

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Theorem # Let A, B be non-empty separated subsets of a top-space and C is connected subset of X. If $C \subseteq A \cup B$, then either $C \subseteq A$ or $C \subseteq B$

Proof # $\because$ A & B are separated subset of X

$\therefore$ $A \cap \bar{B} = \phi$ $\quad \bar{A} \cap B = \phi$

If possible let

$C \cap A \neq \phi$, $C \cap B \neq \phi$

Now

$(C \cap A) \cup (C \cap B) = C \cap (A \cup B)$

$= C \quad \because C \subseteq A \cup B$

$C = (C \cap A) \cup (C \cap B)$

$= S_1 \cup S_2 \qquad S_1 = C \cap A \qquad S_2 = C \cap B$

$\bar{S_1} = \overline{C \cap A} \subseteq \bar{C} \cap \bar{A} \quad \bar{S_2} = \overline{C \cap B} \subseteq \bar{C} \cap \bar{B} \quad [C \cap \bar{C} = C]$

$\bar{S_1} \cap S_2 \subseteq (\bar{C} \cap \bar{A}) \cap (C \cap B)$

$= C \cap (\bar{A} \cap B) = C \cap \phi = \phi$

$\Rightarrow \bar{S_1} \cap S_2 = \phi$

$\bar{S_2} \cap S_1 \subseteq (\bar{C} \cap \bar{B}) \cap (C \cap A)$

$= C \cap (\bar{B} \cap A)$

$= C \cap \phi = \phi$

As such C is disconnected which is a contradiction to hypothesis. Hence

$C \cap A \neq \phi \qquad C \cap B \neq \phi$ both can not hold simultaneously.

So either $C \cap A = \phi$ or $C \cap B = \phi$

either $C \subseteq B$ or $C \subseteq A$ (Proved)

Theorem # (Subspace Criterion)

Y is disconnected subspace of a topological space (X, T) iff there exist non-empty subsets A & B s.t

$Y = A \cup B \qquad (\bar{A})_X \cap B = \phi \qquad A \cap (\bar{B})_X = \phi \quad \rightarrow (1)$

19 + 4 = ...

Proof # Let Y be a disconnected subset of (X, T), Then (Y, T_Y) is a disconnected top-space. Hence there is a pair of non-empty subsets A & B such that

$$Y = A \cup B \qquad (\bar{A})_Y \cap B = \phi \qquad A \cap (\bar{B})_Y = \phi \quad \rightarrow (2)$$

Since $(\bar{A})_Y = (\bar{A})_X \cap Y$, $(\bar{B})_Y = (\bar{B})_X \cap Y$
It follows that

$$(\bar{A})_Y \cap B = \{(\bar{A}_X) \cap Y\} \cap B = (\bar{A})_X \cap \{Y \cap B\}$$
$$= (\bar{A})_X \cap B \qquad \because B \subseteq Y$$

$$A \cap (\bar{B})_Y = A \cap \{(\bar{B})_X \cap Y\} = (A \cap Y) \cap (\bar{B})_X$$
$$= A \cap (\bar{B})_X$$

Thus (2) implies (1)
Conversely let there exists a pair of A, B of non-empty subsets such that

$$Y = A \cup B \qquad (\bar{A})_X \cap B = \phi \qquad (\bar{B})_X \cap A = \phi$$

Since $(\bar{A})_Y \subseteq (\bar{A})_X \qquad (\bar{B})_Y \subseteq (\bar{B})_X$

$$\therefore A \cap (\bar{B})_Y \subseteq A \cap (\bar{B})_X = \phi$$
$$\Rightarrow A \cap (\bar{B})_Y = \phi$$
$$(\bar{A})_Y \cap B \subseteq (\bar{A})_X \cap B = \phi$$
$$\Rightarrow (\bar{A})_Y \cap B = \phi$$

Thus

$$Y = A \cup B \qquad (\bar{A})_Y \cap B = \phi \qquad A \cap (\bar{B})_Y = \phi$$

$\Rightarrow Y$ is disconneted.

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Theorem # Let $(X, \mathcal{T})$ be a top-space and $C \subset X$ be connected. Then $\bar{C}$ is also connected. If $\bar{C}$ is connected, is C necessarily connected.

OR

The closure of a connected subset of a topological space is connected but the converse is not necessarily true.

Proof # If H is any open set and $H \cap \bar{C} \neq \phi$, then $H \cap C \neq \phi$. We first prove it

Let $p \in H \cap \bar{C}$

Then $p \in H$ & $p \in \bar{C}$

$\because$ $p \in \bar{C}$

$\therefore$ Every open set containing p intersect with C and so we have.

$$H \cap C \neq \phi \qquad (\because p \in H \text{ \& } H \text{ is open})$$

Suppose that $\bar{C}$ is not connected and U/V is its disconnection. Then

$$\bar{C} \cap U \neq \phi, \ \bar{C} \cap V \neq \phi, \ U \cap V \cap \bar{C} = \phi$$
$$\bar{C} \subset U \cup V \quad \longrightarrow (1)$$

$\because$ $C \subseteq \bar{C}$, from relation (1), we have.

$$C \cap U \neq \phi \quad C \cap V \neq \phi \quad U \cap V \cap C = \phi$$
$$C \subseteq U \cup V \quad \longrightarrow (2)$$

Relation (2) shows that C is disconnected, a contradiction to the hypothesis. Hence $\bar{C}$ must be connected.

Converse # Let $C = (]0, 1[) \cup (]1, 3[)$; Then C is disconnected and $\bar{C}$ is connected

Metheod-II

Let $\bar{C}$ be disconnected with disconnection U/V as a disconnection.

$\Rightarrow$ $U \cap C / V \cap C$ is a disconnection for C

But C is connected which means

either $U \cap C = \phi$ OR $V \cap C = \phi$

$\Rightarrow$ $C \subseteq U' = V$ OR $C \subset V' = U$

$\Rightarrow$ $\bar{C} \subset \bar{V} = V$ OR $\bar{C} \subset \bar{U} = U$ ($\because U$ & V are closed)

$\Rightarrow$ $\bar{C} \subset V$ OR $\bar{C} \subset U$

$\Rightarrow$ $U = \phi$ OR $V = \phi$

$\Rightarrow$ U/V is not a proper disconnection but trivial.

$\Rightarrow$ $\bar{C}$ is not disconnected but connected.

Theorem # Let X be a topological space and A a connected subspace of X. If B is a subspace of X such that $A \subseteq B \subseteq \bar{A}$, then B is connected, in particular $\bar{A}$ is connected. ($A \subseteq B \subseteq \bar{A}$ means B is is dense in $\bar{A}$)

Proof # Assume that B is disconnected and have a disconnection. $G \cup H$ i.e

$B \subset G \cup H$ $\quad B \cap G \neq \phi$ $\quad B \cap H \neq \phi$

$B \cap G \cap H = \phi$

$\because$ $A \subset B$

$\therefore$ $A \subset G \cup H$

Since A is a connected subset of a disconnected set B with disconnection G/H, therefore we have

$A \subseteq G$ or $A \subseteq H$ [Sep- Son-rule]

Now $A \subseteq G \Rightarrow \bar{A} \subseteq \bar{G}$

$\Rightarrow \bar{A} \cap H \subseteq \bar{G} \cap H = \phi$

$\Rightarrow \bar{A} \cap H = \phi$

$\because$ $B \subseteq \bar{A}$

$\therefore$ $B \cap H = \phi$ which is a contradiction

Similarly $A \subseteq H \Rightarrow \bar{A} \subseteq \bar{H}$

$\Rightarrow \bar{A} \cap G \subseteq \bar{H} \cap G = \phi$

$\Rightarrow \bar{A} \cap G = \phi$

So $B \cap G = \phi$ which is a contradiction

OR

{We can get a contradiction as follows.}
{we have proved that $\bar{A} \cap H = \phi$}

Assume that B is disconnected, and G/H is a disconnection such that G, H are separated sets i.e

$B = G \cup H \qquad \bar{G} \cap H = \phi \qquad G \cap \bar{H} = \phi$

$\because$ A is a connected subset of a disconnected set B

$\therefore \quad A \subseteq G$ or $A \subseteq H$

(i) $A \subseteq G \Rightarrow \bar{A} \subseteq \bar{G} \Rightarrow \bar{A} \cap H \subseteq \bar{G} \cap H = \phi$

$\Rightarrow \bar{A} \cap H = \phi \rightarrow ①$

(ii) Also $H \subseteq G \cup H = B \subseteq \bar{A}$

$H \subseteq \bar{A}$

$H \cap H \subseteq \bar{A} \cap H$

$H \subseteq \bar{A} \cap H$

But $\bar{A} \cap H \subseteq H$ by definition

$\Rightarrow \bar{A} \cap H = H$

But $\bar{A} \cap H = \phi$ from (i)

So $H = \phi$

$\Rightarrow B = G \cup \phi = G \Rightarrow$ B has no partition

Similarly $A \subseteq H \Rightarrow B = H \Rightarrow$ B has no partition

These contradictions show that our supposition is wrong and B is connected.

<u>Deduction I</u> $\because A \subseteq \bar{A} \subseteq \bar{A}$

Therefore $\bar{A}$ is connected.

<u>Deduction II</u> Deduce that semi-open intervals in R are connected.

The half-onpen intervals $[a\ b[$ & $]a\ b]$ are connected in R because

$]a\ b[\subseteq [a\ b[,\ ]a\ b] \subseteq [a\ b]$

$C \subseteq [a\ b[,\ ]a\ b] \subseteq \bar{C}$

Hence by above theorem $[a\ b[,\ ]a\ b]$ are connected in R

Theorem # Let $(X.T)$ be a metric space and $C \subset X$. Prove that $\bar{C}$ is Connected iff
$C \neq A \cup B$, $A \neq \phi$ $B \neq \phi$ Such that $\bar{A} \cap \bar{B} = \phi$

Proof # Let $\bar{C}$ be Connected. and if possible let

$C = A \cup B$, $A \neq \phi$ $B \neq \phi$, $\bar{A} \cap \bar{B} = \phi$

Now

$\bar{C} = \overline{A \cup B} = \bar{A} \cup \bar{B}$ $\quad (\because \overline{A \cup B} = \bar{A} \cup \bar{B})$

$(\bar{\bar{A}}) \cap \bar{B} = \bar{A} \cap \bar{B} = \phi$

$(\bar{A}) \cap (\bar{\bar{B}}) = \bar{A} \cap \bar{B} = \phi$.

This shows that $\bar{C}$ has a disconnection. $\bar{C} = \bar{A} \uplus \bar{B}$. This obviously contradicts the connectedness of $\bar{C}$. Hence C cannot be expressed as union of two non-empty subsets A & B such that $\bar{A} \cap \bar{B} = \phi$.
Conversely if $C \neq A \cup B$, $A \neq \phi$ $B \neq \phi$, $\bar{A} \cap \bar{B} = \phi$ Then C is not the union of two non-empty separated sets. This means C is Connected.
As closure of a Connected Set is Connected, it follows that $\bar{C}$ is Connected

Theorem # Let A & B be two closed sets in a top-space $(X.T)$. Show that if $A \cap B$ & $A \cup B$ are Connected, then A is connected.

Proof # if $A \subseteq B$, then $A \cap B = A$

$\because A \cap B$ is connected $\therefore A$ is Connected.

If $B \subseteq A$, then $A \cup B = A$

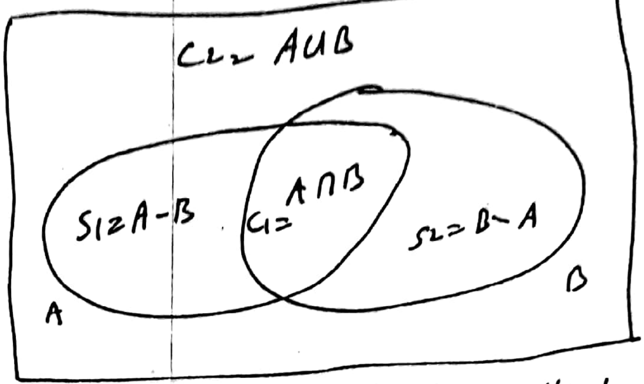
$\because A \cup B$ is Connected $\therefore A$ is Connected

Thus when $A \subseteq B$ or $B \subseteq A$, the set A is Connected.
Now suppose that $A \nsubseteq B$ & $B \nsubseteq A$. Then

$S_1 = A \setminus B \neq \phi$ $\quad S_2 = B \setminus A \neq \phi$

$S_1 = C_2 - B$

$S_2 = C_2 - A$



Since A & B are closed sets, it follows that S_1 & S_2 are open sets in $C_2 = A\cup B$

Further $S_1 \cap S_2 = \phi$

$\Rightarrow S_1$ & S_2 are separated sets

Now $C_1 = A\cap B$ is connected by hypothesis

$C_1' = C_2 - C_1 = S_1 \cup S_2$ (union of two separated sets)

It follows that $C_1 \cup S_1 = A$ is connected set because if C is a connected set and $C' = S_1 \cup S_2$ (S_1, S_2 are separated sets), then $C\cup S_1$ & $C\cup S_2$ are connected sets.

Theorem # The union of two non-empty separated sets in a top-space is disconnected.

Proof # Let A & B be two non-empty separated sets, then

$$A\cap \bar{B} = \phi \qquad \bar{A}\cap B = \phi \quad \to (1)$$

Let $D = A\cup B$

Let $G = X - \bar{B} = (\bar{B})'$ $\quad H = X - \bar{A} = (\bar{A})'$

Then G & H are open because $\bar{A}, \bar{B}$ are closed.

Now

$$D_1 = D\cap G = (A\cup B)\cap(\bar{B})'$$
$$= \{A\cap(\bar{B})'\}\cup\{B\cap(\bar{B})'\}$$
$$= A\cup\phi \qquad \because A\cap\bar{B} = \phi \Rightarrow (\bar{B})' \supseteq A$$
$$= A \neq \phi \qquad \text{Also } \because B\subseteq\bar{B}, \ \therefore B\cap(\bar{B})' = \phi$$

Similarly $D_2 = D \cap H = B \neq \phi$

$D_1 \cap D_2 = A \cap B = \phi$

$D_1 \cup D_2 = A \cup B = D$

Thus $\{D_1, D_2\}$ forms a disconnection of D and hence $D = A \cup B$ is disconnected.

Theorem # Prove that following statements are equivalent

A subset C of a top-space disconnected iff

(i) There exist two non-empty open sets G & H such that

$C \cap G \neq \phi \quad C \cap H \neq \phi \quad C \cap G \cap H = \phi$

$C \subseteq G \cup H$

(ii) There exist two non-empty sets A & B s.t

$A \cap \bar{B} = \phi \quad \bar{A} \cap B = \phi \quad C = A \cup B$ i.e

C is union of two separated sets.

Proof # Let the statement (i) be true about subset C.

Then

let $A = C \cap G \neq \phi \quad B = C \cap H \neq \phi$.

$A \cap B = (C \cap G) \cap (C \cap H)$

$= C \cap (G \cap H)$

$= \phi$ (given)

$A \cup B = (C \cap G) \cup (C \cap H)$

$= C \cap (G \cup H)$

$= C \quad \because C \subseteq G \cup H$

Now we prove that A & B are separated i.e

$A \cap \bar{B} = \phi \qquad \bar{A} \cap B = \phi$

Let $x \in B^d$ and $x \in A = C \cap G$

$\Rightarrow x \in G$.

Since x is a limit point of B and G is an open set

Containing x, by definition of limit point G must contain a point of B other than x i.e

$$B \cap G \neq \phi$$

$$C \cap H \cap G \neq \phi$$

But $C \cap H \cap G = \phi$ (given)

a Contradiction.

Hence A contains no limit point of B i.e

$$A \cap \bar{B} = \phi$$

Similarly $\bar{A} \cap B = \phi$

Thus Condition (ii) is proved.

(ii) $\rightarrow$ (i)

Let the condition (ii) be true for set C i.e $\exists$ two non-empty separated sets A & B s.t

$A \cap \bar{B} = \phi$ $\quad \bar{A} \cap B = \phi$ and $C = A \cup B$

We prove condition (i)

Let $P = C \cap G \neq \phi$, $\quad Q = C \cap H \neq \phi$

where $G = X - \bar{A} = (\bar{A})'$ open

$H = X - \bar{B} = (\bar{B})'$ open

Then $C \cap G = (A \cup B) \cap (\bar{A})'$

$$= \{A \cap (\bar{A})'\} \cup \{B \cap (\bar{A})'\}$$

$$= \phi \cup B = B \neq \phi$$

$\because \bar{A} \cap B = \phi \Rightarrow (\bar{A})' \supseteq B$

Also $A \subseteq \bar{A} \Rightarrow A \cap (\bar{A})' = \phi$

$A \cap \bar{B} = \phi \Rightarrow (\bar{B})' \supseteq A$

$$C \cap H = (A \cup B) \cap (\bar{B})'$$

$$= \{A \cap (\bar{B})'\} \cup \{B \cap (\bar{B})'\}$$

$$= A \cup \phi = A \neq \phi$$

$$C = A \cup B$$

$$= (C \cap H) \cup (C \cap G)$$

$$= C \cap (H \cup G)$$

$$\Rightarrow C \subseteq H \cup G$$

Also $C \cap G \cap H = (C \cap G) \cap H$

$$= B \cap H = B \cap (\bar{B})' = \phi.$$

Thus we have obtained two open sets G, H s.that

$C \cap G \neq \phi$ $C \cap H \neq \phi$ $C \subseteq G \cup H$
and $C \cap H \cap G = \phi$
which is our Condition (i)

Theorem # A top-space X is disconnected iff there two non-empty subsets A & B such that

$X = A \cup B$ $A \cap B = \phi$
$A \cap \bar{B} = \phi$ $\bar{A} \cap B = \phi$

Proof # Let X be disconnected. Then there exist two non-empty closed sets A & B such that
$X = A \cup B$ $A \cap B = \phi$
$\because$ A & B are both closed.
$\therefore$ $\bar{A} = A$ $\bar{B} = B$
and therefore
$A \cap \bar{B} = A \cap B = \phi$
$\bar{A} \cap B = A \cap B = \phi$
So condition is proved.
Conversely if the given condition is true. Then
$A \cap \bar{B} = \phi$ & $\bar{A} \cap B = \phi$
Now $A \cap \bar{B} = \phi$ $\because X = A \cup B$
$\Rightarrow \bar{B} \subseteq A^c = B$ $B^c = A$, $A^c = B$
But $B \subseteq \bar{B}$
$\Rightarrow \bar{B} = B$
$\Rightarrow B$ is closed.
$\bar{A} \cap B = \phi$ $\Rightarrow$ $\bar{A} \subseteq B^c = A$
But $A \subseteq \bar{A}$
$\Rightarrow A = \bar{A}$
$\Rightarrow A$ is closed.

Hence there are two-non-empty disjoint closed sets A & B such that
$X = A \cup B$. $\Rightarrow X$ is disconnected

Theorem # Let $\mathcal{C} = \{C_\alpha : \alpha \in I\}$ be a collection of connected sets. If no two members of $\mathcal{C}$ are separated, then $\bigcup_{\alpha \in I} C_\alpha$ is connected.

Proof # Let $E = \bigcup_{\alpha \in I} C_\alpha$ be disconnected.

with G & H as disconnection.

$\because$ each C_α is a subset of a disconnected set E with disconnection $\{G, H\}$

$\therefore$ either $C_\alpha \subseteq G$ or $C_\alpha \subseteq H$ $\quad \forall \alpha \in I$

Now any two members say C_λ, C_v are not separated, so

$$C = C_\lambda \cup C_v \text{ is connected.}$$

Now C is a connected subset of a disconnected set E with disconnection G/H. Therefore

$$C \subseteq G \quad \text{or} \quad C \subseteq H$$

and C is disjoint from the other

Similarly we can prove that all the members of $\mathcal{E}$ and hence E must be contained in G or in H and disjoint from the other

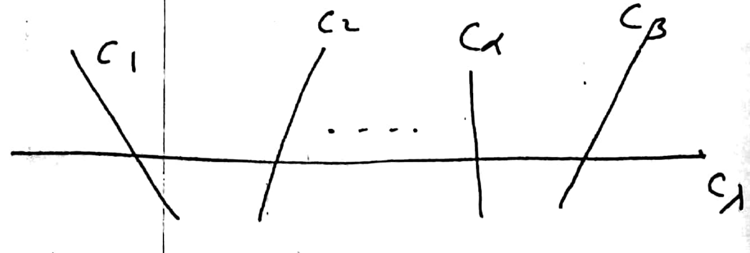
Let $E \subseteq G$ $\quad$ Then $E \cap H = \phi$

which is a contradiction.

Hence E is connected.

Theorem # Let $\{C_\alpha : \alpha \in I\}$ be a family of connected subsets of a top-space. If a particular number C_λ of the family intersects every other member, show that $\bigcup\{C_\alpha : \alpha \in I\}$ is connected.

Proof #



Proof # Let $C = \bigcup\{C_\alpha : \alpha \in I\}$.
We are given that

$$C_\lambda \cap C_\alpha \neq \phi \qquad \forall \alpha \in I.$$

We are to prove that C is connected.
Let

$$F_\alpha = C_\lambda \cup C_\alpha \qquad \alpha \in I$$

$\because$ F_α is union of two non-disjoint connected sets
$\therefore$ F_α is connected for each $\alpha \in I$

$$\bigcup_{\alpha \in I} F_\alpha = \bigcup_{\alpha \in I}(C_\lambda \cup C_\alpha)$$

$$= \left(\bigcup_\alpha C_\alpha\right) \cup C_\lambda$$

$$= \bigcup_\alpha C_\alpha \qquad \because C_\lambda \subseteq \bigcup_\alpha C_\alpha$$

For $\alpha \neq \beta$, $\alpha, \beta \in I$
we have.

$$F_\alpha \cap F_\beta = (C_\lambda \cup C_\alpha) \cap (C_\lambda \cup C_\beta)$$

$$= C_\lambda \cup (C_\alpha \cap C_\beta)$$

$$\neq \phi \qquad C_\lambda \neq \phi$$

It follows that $\{F_\alpha : \alpha \in I\}$ is the class of connected sets with non-empty intersection.
Hence $\bigcup\{F_\alpha : \alpha \in I\}$ is connected. But

$$\bigcup_\alpha C_\alpha = \bigcup_\alpha F_\alpha$$

$\Rightarrow$ $C = \bigcup_\alpha C_\alpha$ is also connected (Proved)

Theorem # A subset A of a top-space is connected iff A is not the union of two non-empty separated sets.

Proof # We prove that A is disconnected iff A is the union of two-non-empty separated sets.

Suppose that A is disconnected and $G \cup H$ be the disconnection of A. Then

$(A \cap G) \cup (A \cap H) = A$ and $(A \cap G) \cap (A \cap H) = \phi$

Now we prove that $A \cap G$, $A \cap H$ are separated.

let $P = A \cap G$ $\quad Q = A \cap H$

We are to prove that

$$P \cap \bar{Q} = \phi \qquad \bar{P} \cap Q = \phi$$

Let $x \in P^d$ and $x \in Q = A \cap H$. Then.

$x \in H$, H is open. By definition of limit point of P, H must contain a point of P other than x i.e

$$P \cap H \neq \phi$$

$$(A \cap G) \cap H \neq \phi$$

$$\Rightarrow A \cap G \cap H \neq \phi$$

But $G \cup H$ is a disconnection of A and $A \cap G \cap H = \phi$. So we have a Contradiction.

Hence $\bar{P} \cap Q = \phi$

Similarly we can prove that

$$P \cap \bar{Q} = \phi$$

Thus A is union of two non-empty separated sets.

Conversely let $A = G \cup H$ such that

$$G \cap \bar{H} = \phi \qquad \bar{H} \cap G = \phi$$

Now union of non-empty separated sets is disconnected. Hence $G \cup H$ is disconnected and therefore A is disconnected.

Hence the Result That

A is connected iff it not union of two non-empty separated sets.

Connected Points in a Top- Space

Two points a, b of a top- space X are said to be connected if they are contained in a connected subset of X.

Theorem # Let D be a subset of a top- space X. If every two points in set D are in some connected subset C of D, D must be connected. Deduce connectedness of (X T).

OR

If every pair of points in set D is connected, then D must be connected.

Proof # Assume that D is disconnected. Let A, B be two non- empty separated sets such that

$A \cup B = D$ $\quad\quad \bar{A} \cap B = \phi \quad A \cap \bar{B} = \phi$

Since $A \neq \phi$ $\quad B \neq \phi$

Let $a \in A$ $\quad b \in B$

Then $a \neq b$ because $A \cap B = \phi$

$\Rightarrow$ a, b are distinct points of D

By hypothesis $a, b \in C$, where C is some connected subset of D.

$\because$ D is disconnected with A/B as disconnection and C is a connected subset of D

$\therefore$ $C \subseteq A$ or $C \subseteq B$

This is not possible because $A \cap B = \phi$ and C contains at least one point (a) of A and at least one point (b) of B. Hence a contradiction

Thus D must be connected.

Deduction # Idetify D with X to obtain the result (X.T) is connected if each pair of points in X is

is contained in a connected subset C of X.

Theorem # A top-space X is connected iff for every pair of points in X, there is some connected subspace (subset) which contains both i.e if every pair of point in X is connected.

Proof # Let X be a connected space and x, y be two points in X. Then Then there is a connected subspace namely X itself which contains both x & y.

Conversely suppose that for any two points x, y in X, there is a connected sub-space $C_{x,y}$ of X which contains both x & y.

For any fixed point a in X consider the collection

$\{ C_{a,x} : x \in X \}$ of all connected subspaces of X which contain both a & x. Then

$$X = \bigcup_{x \in X} C_{a,x} \quad \text{and} \quad \bigcap_{x \in X} C_{a,x} \neq \phi$$

Hence X is connected because union of connected subsets with non-empty intersection is connected.

OR

Assume that X is disconnected with A/B disconnection such that

$$X = A \cup B \qquad \bar{A} \cap B = \phi \qquad A \cap \bar{B} = \phi$$

Let $a \in A$, $b \in B$, then $a \neq b$

By hypothesis $\exists$ a connected subset C such that $a, b \in C$

$\because$ C is a Connected subset of a disconned set X

$\therefore$ $C \subseteq A$ or $C \subseteq B$

This is not possible because $A \cap B = \phi$ and C contains at least one element of A and at least one element of B. This contradiction leads us to Connectedness of X

Theorem # (Key Criterion for disconnectedness or Double - edge rule)

A top-space is disconnected iff there exist non-empty proper subset A which is both open and closed.

Proof # Let $A \neq \phi$ be a proper subset of X which is both open and closed. Then since A is proper subset of X, $A' \neq \phi$.

Also A' is both open and closed and

$A \cap A' = \phi$ $\qquad \bar{A} = A \qquad \bar{A'} = A'$

$\bar{A} \cap A' = A \cap A' = \phi$

$A \cap \bar{A'} = A \cap A' = \phi$

Also $X = A \cup A'$

It follows that X is union of two non-empty separated sets A & A'. Hence X is disconnected.

Conversely let X be a disconnected top-space

There exist subsets $A \neq \phi$ $\quad B \neq \phi$ such that

$X = A \cup B$ $\qquad A \cap \bar{B} = \phi \qquad \bar{A} \cap B = \phi$ $\rightarrow (1)$

$A' = B$ $\qquad B' = A$

$\bar{A} \cup B \subseteq X$ $\qquad \because A \subseteq \bar{A}$

and $A \cup B \subseteq \bar{A} \cup B$

$\Rightarrow X \subseteq \bar{A} \cup B$

Thus $\bar{A} \cup B = X$ Similarly $A \cup \bar{B} = X$ $A' \cap B = \phi$ $\rightarrow$ ②

Using ① & ② we obtain the following

$$A \cup B = X \quad A \cap B = \phi \Rightarrow A = X - B = B' \rightarrow ③$$
$$A \cup \bar{B} = X \quad A \cap \bar{B} = \phi \Rightarrow A = X - \bar{B} = (\bar{B})' \rightarrow ④$$
$$\bar{A} \cup B = X \quad \bar{A} \cap B = \phi \Rightarrow B = X - \bar{A} = (\bar{A})' \rightarrow ⑤$$

$\because$ $A \neq \phi$ & $B \neq \phi$

$\therefore$ It follows from ③ that A is a proper subsets of X

Result ④ shows that A is open

Result ③ & ⑤ show that gives

$$B = (\bar{A})'$$
$$B' = \bar{A}$$

putting $B' = A$ from ③

$$A = \bar{A}$$

$\Rightarrow$ A is closed.

Thus A is a proper subset of X, which both open and closed.

<u>OR</u>

Let X be disconnected. Then $\exists$ two non-empty open sets A & B such that

$$X = A \cup B \qquad A \cap B = \phi$$

$\Rightarrow A' = B$ (open)

$\Rightarrow$ A is closed

Thus A is a proper subset of X which is both open and closed.

<u>Corollary#</u> A ~~top~~ space $(X, \mathcal{T})$ is connected iff the only non-empty subset of X which is both open and closed is X itself. i.e iff X does not contain a non-emty proper subset which is both open and closed.

P.T.O

Proof# Let (X, T) be connected. Suppose $A \neq \phi$ is a subset of X which is both open and closed. As such $A' \neq \phi$ is both open and closed. Thus

$\bar{A} = A$ $\quad (\bar{A'}) = A'$

$A \cap A' = \phi$ $\quad X = A \cup A'$

Now

$A \cap A' = \phi \Rightarrow \bar{A} \cap A' = \phi$ $\quad \because A' \subseteq (A')'$ $\rightarrow A' \subseteq A$

because if $x \in A^d$, $x \in A'$, then A' is an open set contain a limit pnt of A and A' must contain a pnt of A other than x which means that

$A \cap A' \neq \phi$

which is a contradiction. Hence $\bar{A} \cap A' = \phi$

Similarly $\quad A \cap \bar{A'} = \phi$

It follows A, A' are two separated sets with

$X = A \cup A'$

Now if $A \neq \phi$, $A' \neq \phi$

Then separation $X = A \cup A'$ implies that X is disconnected which contradicts the hypothesis that X is connected. Hence

either $A = \phi$ or $A' = \phi$

Since $A \neq \phi$, therefore $A' = \phi$

Hence $X = A \cup A' = A \cup \phi = A$

It follows that the only non-empty set which is both open and closed is X itself.

Converse Let the only non-empty set which is both open and closed is X itself i.e X contains no proper subset which is both open and closed. Then by above theorem X is connected space.

OR

Let X be disconnected and $\{A, B\}$ be a disconnection of X where $A \cup B = X$ and $A \cap B = \phi$, A & B are both open

$$A \cap B = \phi$$
$$\Rightarrow A \cap B = B' \cap B$$
$$\Rightarrow A = B'$$

$\because$ B' is closed $\therefore$ A is closed

$A \neq \phi$ $B \neq \phi$ $\Rightarrow$ A is a proper subset of X or $X \neq A$.

Thus A is a proper subset of X which is both open and closed. This is a contradiction to the hypothesis. Hence X is connected.

Comment # Let (X.T) be a top-space and (Y, T_Y) be its subspace. Then Y is connected iff the only subset of Y which is both open and closed in X is Y itself

Connectedness being an absolute property, Y is connected iff (Y, T_Y) is connected and hence by Corollary the result follows.

Theorem # A top-space is disconnected iff X is the union of two non-empty disjoint open (closed) sets in (X.T).

Proof (a) Let the space (X.T) be disconnected. Then ∃ a non-empty proper subset A of X that is both open and closed. As such A' is non-empty subset of X that is both open and closed. obviously

$$X = A \cup A' \qquad A \cap A' = \phi$$

Thus X is union of two non-empty disjoint open sets. or (closed sets)

Converse # Let $X = A \cup B$ $A \neq \phi$ $B \neq \phi$, A open, B open

$\because$ $X = A \cup B \Rightarrow B' = A$

as B is open, A is closed. As $B \neq \phi$, it follows.

$A \neq \phi$ is a proper subset of X which is both open and closed. Hence X is disconnected.

(b) Case of closed sets Interchange the words close and open.

Corollary# Let $(X.T)$ be a top-space and let $Y \subset X$ (proper subset). Then Y is disconnected iff there exist two non-empty subsets G & H open (or closed) in X such that

$$Y \cap G \neq \phi \qquad Y \cap H \neq \phi \qquad Y \cap (G \cap H) = \phi$$

$$Y \cap (G \cup H) = (Y \cap G) \cup (Y \cap H) = Y$$

Proof# Let (Y, T_Y) be disconnected. Then Y is union of two non-empty disjoint open sets G_1 & H_1 (say) in Y i.e

$$G_1 \neq \phi \qquad H_1 \neq \phi \qquad G_1 \cap H_1 = \phi \qquad G_1 \cup H_1 = Y$$

This is true iff there exist open sets G, H in X such that

$$Y \cap G \neq \phi \qquad Y \cap H = \phi \qquad (Y \cap G) \cap (Y \cap H) = \phi$$

$$(Y \cap H) \cup (Y \cap G) = Y$$

$$\Leftrightarrow \quad Y \cap G \neq \phi \qquad Y \cap H \neq \phi \qquad Y \cap (G \cap H) = \phi$$

$$Y \cap (G \cup H) = Y$$

Comments # (i) $Y \cap (G \cup H) = Y$ may be replaced by $Y \subseteq G \cup H$

(ii) G & H may be replaced by closed sets E & F.

Theorem # A subset A of a top-space $(X.T)$ is open and closed in X iff $A^b = \phi$ (boundary of $A = \phi$)

Proof# Let A be open and closed in X

$x \in A^b$. By definition of boundry.

$$A^b = \bar{A} \cap \bar{A'}$$

$\Rightarrow x \in \bar{A} \cap \bar{A'}$

$\Rightarrow x \in \bar{A}$ & $x \in \bar{A'}$

$\because$ A is closed

$\therefore \bar{A} = A$

$\Rightarrow x \in A$

$\because$ A is open

$\therefore$ A' is closed and $\bar{A'} = A'$

So $x \in A'$

Hence $x \in A \cap A' = \phi$

which is a Contradiction. Hence A has no boundry

Conversely let $A^b = \phi$. We are to prove that A is both open and closed.

$\because A^b = \phi$

$\Rightarrow \bar{A} \cap \bar{A'} = \phi$

If p is a limit point of A, $p \in \bar{A}$ and hence

$p \notin \bar{A'}$

$\Rightarrow p \notin A'$

Thus $p \in A$ and A is closed.

Similarly let q be limit point of A', then $q \in \bar{A'}$ and hence

$q \notin \bar{A}$

$\Rightarrow q \notin A$

$\Rightarrow q \in A'$ and A' is closed.

$\Rightarrow$ A is open.

Hence A is both open and closed.

<u>Theorem#</u> If a Connected set C in a top-space has a non-empty intersection with a set A and its Complement in top-space, then C has a non-empty intersection with boundry of A i.e $C \cap A^b \neq \phi$

Deduce that in a Connected top-space. Every proper subset $A \neq \phi$ has a non-empty boundry

Proof # Assume that

$$C \cap A^b = \phi$$

Now

$$C = C \cap X = C \cap (A \cup A^c)$$
$$= (C \cap A) \cup (C \cap A^c)$$
$$= C_1 \cup C_2$$

By hypothesis

$$C_1 = C \cap A \neq \phi \qquad C \cap A^c \neq \phi$$

Also $C_1 \subseteq A$, $C_2 \subseteq A^c$

So $\bar{C_1} \subseteq \bar{A}$, $\bar{C_2} \subseteq \bar{A^c}$

Now using closure properties & $A^b = A \cap \bar{A'}$

We get

$$C_1 \cap \bar{C_2} = \{C \cap A\} \cap \{\overline{C \cap A'}\}$$
$$\subseteq \{C \cap A\} \cap \{\bar{C} \cap \bar{A'}\}$$
$$\subseteq C \cap A \cap \bar{A'} = C \cap A^b = \phi$$

Similarly $\bar{C_1} \cap C_2 = \phi$

Consequently, C has a separation C_1 / C_2 and so C is dis-connected which is Contrary to hypothesis. Thus the assumption $C \cap A^b = \phi$ stands rejected. and $C \cap A^b \neq \phi$

Deduction # here $(X\ T)$ is a Connected top-space and $X \cap A \neq \phi$, $X \cap A' \neq \phi$. Hence by above theorem we deduce that

$$X \cap A^b \neq \phi$$
$$\Rightarrow A^b \neq \phi$$

Theorem # Show that a topological space is Connected iff every non-empty proper subspace (subset) has a non-empty boundry

Proof # Suppose that a top-space X is connected. We are to prove that a non-empty proper subspace A of X has a non-empty boundry.

On the contrary suppose that a non-empty proper subset A of X has an empty boundry i.e $A^b = \phi$

Then A is open and (A has no boundry so open)

$$A' = X - A$$

So

$$\bar{A} \cap \overline{A^c} = \phi$$

Let p be a limit point of A, then $p \in \bar{A}$ but $p \notin \overline{A^c}$

In particular $p \notin A^c$ because p is not limit point of A^c $(\because p \notin A^c \cup (A^c)^d)$

So $p \in A$

$\ni$ A contains all of its limit points. Hence A is closed

Thus A is a proper subset of X which is both open and closed. Consequently X is connected, a contradiction. Hence $b(A) \neq \phi$

Conversely suppose that every non-empty proper subspace A of X has a non-empty boundry. We prove that X is connected.

Suppose that X is disconnected. Then there is a proper subset A of X which is both open and closed. But then

$\bar{A} = A$ $\qquad \overline{A^c} = A^c$ $\quad \because A^c$ is closed

and $\bar{A} \cap \overline{A^c} \neq \phi$ and $\bar{A} \cup \overline{A^c} = X$

becaus $\bar{A} \cap \overline{A^c} = A \cap A^c = \phi$

and $\bar{A} \cup \overline{A^c} = A \cup A^c = X$

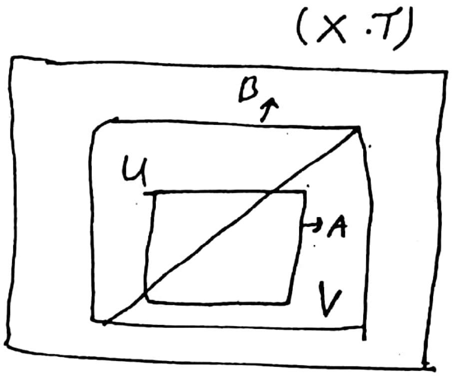
So $\{A, \overline{A^c}\}$ is a disconnection of X, a contradiction.

Thus X is connected.

Theorem # Let $(X.T)$ be a top- space and. A, B be two subspaces s.t $A \subseteq B$. If A is connected, then B is Connected.

Proof # Let B be disconnected and $\{U, V\}$ be a disconnection for B s.t

$U \cap V = \phi$ $\quad U \cup V = B$



$\because A \subseteq B$

$\therefore \{U \cap A, V \cap A\}$ is a disconnection for A s.t

$(U \cap A) \cap (V \cap A) = \phi$.

$(U \cap A) \cup (V \cap A) = A$

Since A is Connected subset of itself, Therefore Either $U \cap A = \phi$ or $V \cap A = \phi$

If $U \cap A = \phi$, Then $A \subseteq U' = V$

$\Rightarrow A \subseteq V \Rightarrow \bar{A} \subseteq \bar{V} = V$.

If $V \cap A = \phi$, Then

$A \subseteq V' = U \Rightarrow A \subseteq U \Rightarrow \bar{A} \subseteq \bar{U} = U$

$B \subseteq \bar{A} = U \Rightarrow B \subseteq U \Rightarrow V = \phi$ { Every

$B \subseteq \bar{A} = V \Rightarrow B \subseteq V \Rightarrow U = \phi$

$\{U, V\}$ is not a proper discotion for $\Rightarrow B$ is Conneded.

Theorem # Prove that the closed interval $I = [a\ b] \subset R$ is Connected

OR

Prove that intervals are connected subspaces of R.

Proof # Let $I = [a\ b]$ be not Connected. Then $\exists$ open sets $G.H$ in R. s.that

S. 1

$G \cap I \neq \phi$ $\quad H \cap I \neq \phi$

$I \cap G \cap H = \phi$ $\quad I \subseteq G \cup H$

Let $b \in H$ and $m = \sup(G \cap I)$, which exists because bounded above.

Now

$(G \cap I)' = G' \cup I' = H \cup I' =$ union of open sets $\Rightarrow G \cap I$ is closed

$\Rightarrow m \in G \cap I$ $\quad \because G \cap I$ is closed

Now $m \neq b$ $\quad$ Since $m \notin H$ and $b \in H$ ∴

Any nbd of m meets $H \cap I$ Since $m \neq b$ and no neighd of m can be entirely contained in G since $m = \sup(G \cap I)$. Thus m is an accumulation point of $H \cap I$

Similarly $H \cap I$ is closed, so $m \in H \cap I$. This contradicts the fact that $G \cap H \cap I = \phi$

Hence I must be connected.

<u>OR</u>

Let $I = A \cup B$, where A & B are both closed in I. Let $b \in B$. Since A is a bounded set of real numbers, if $A \neq \phi$, then A must have supremum c and $a \leq c \leq b$. Since A is closed, $c \in A$.

If $c = b$, then $c \in B$, $\quad A \cap B \neq \phi$

if $c \leq b$, then $S = \{x : c \leq x \leq b\} \subseteq B$, by definition of c.

As B is closed, $c \in B$. Thus $c \in A \cap B$. This means no separation of I can exist and hence I must be connected

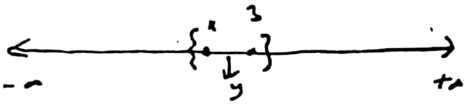
Theorem # A subset (subspace) of the real line R containing at least two points is connected iff it is an interval. In particular R is connected.

Proof # Let X be a connected subspace of R. We prove that X is an interval.

Suppose that X is not interval. Then $\exists$ real numbers x, y, z such that

(i) $x < y < z$ (ii) $x, z \in X$, $y \notin X$

$X = [X \cap (-\infty, y)] \cup [X \cap (y, +\infty)]$



$\Rightarrow$ $[X \cap (-\infty, y)] \cup [X \cap (y, +\infty)]$ is a disconnection of X. So X is a disconnected which is a contradiction. Hence X is an interval

Conversely suppose that X is an interval, we show that X is connected. Suppose that X is disconnected with $G \cup H$ as disconnection.

$\because$ G & H are non-empty and $G \cap H = \phi$

$\therefore$ we can choose $x \in G$, $z \in H$ such that

$x \neq z$

Since X is an interval, $[x, z] \subseteq X$ and each point of $[x, z]$ is either in G or H.

Let $y = \sup\{[x, z] \cap G\}$

It is clear that

$x \leq y \leq z \Rightarrow y \in X$ ($\Rightarrow$ because z is an upper bound for $[x, z] \cap G$)

Since G is closed, $y = \sup([x, z] \cap G) \Rightarrow y \in G$

from this we conclude that $y < z$ $(\because G \cap H = \phi)$

Again from $y = \sup\{[x, z] \cap G\}$, it follows that for $\epsilon > 0$ if

$y + \epsilon \leq z$, then $y + \epsilon \in H$ because $G \cap H = \phi$

and y is l.u.b. Consider the set $\{y+\epsilon, \epsilon > 0, y+\epsilon \leq z, y+\epsilon \in H\}$

which is a subset of H. Now every open set (open interval) of R containing y will contain a point of of this set and hence of H. So y is a limit point of H. Since H is closed, $y \in H$

So $y \in G \cap H \Rightarrow G \cap H \neq \phi$.

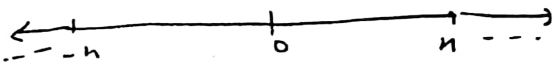
which is a contradiction. Hence X is connected.

In particular $R = (-\infty, +\infty)$ is an interval, Therefore R is connected. OR can be prove as given in example below

Example # The real line R with with usual topology is connected.

Sol # For each +ve integer n, the closed interval $J_n = [-n, n]$ is connected. Also $0 \in J_n$ $\forall$ n +ve integer

$\Rightarrow \bigcap_n J_n \neq \phi$.



Consequently $R = \bigcup_n J_n$ is connected.

Comments # Connectedness is not a hereditary property

Support # The real line R with usual topology is connected. But Q (set of rational numbers) as a subspace of R is not connected because Q is not an interval i.e Q does not inherit the property of connectedness from its parent set R.

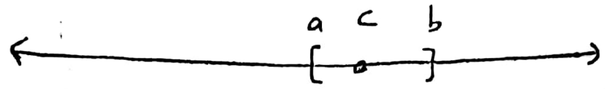
Theorem # A subset X of R is connected iff whenever $a, b \in X$ with $a < b$, we have $[a, b] \subset X$

Proof # Let X be connected subset of R. Suppose there exist $a, b \in X$ with $a < b$, but $c \notin X$ for some $c \in [a\ b]$.

Then $U = \{x \in X \mid x < c\}$

and

$V = \{x \in X \mid c < x\}$



$X = \{a, b \ldots\}$

are non-empty subset of X such that

$\because X = U \cup V$ and $U \cap V = \phi$

$\Rightarrow X$ is not connected and we have a contradiction

Thus whenever $a, b \in X$ with $a < b$, we have $[a . b] \subset X$

Now suppose that whenever $a, b \in X$ with $a < b$, we have $[a . b] \subseteq X$. Let X be not connected with disconnection $\{G\ H\}$, where G, H are non-empty closed sets such that

$X = G \cup H$ $\quad G \cap H = \phi$

Let $a \in G$ and $b \in H$

We assume that $a < b$.

Let $c = \text{lub}\{x \in G \mid x < b\}$

For every +ve integer n, there exist $x_n \in G$ such that $c - 1/n < x_n \leq c$

Thus $\lim_{n \to \infty} x_n = c$ and c is a limit point of G

Since G is closed, $c \in G$

We next show that $(c . b) \subseteq H$. If there exis $y \in (c . b)$ such that $y \notin H$, $y \in G$.

Now $y < b$ and $y \in G$. so $y \in \{x \in G \mid x < b\}$

Thus

$y \leq c = \text{Sup}\{x \in G \mid x < b\}$

$\because y \in (c . b)$, $y > c$ which is impossible. Hence

$(c, b) \subseteq H$

Because $(c, b) \subseteq H$, c is a limit point of H and since H is closed, $c \in H$. But now
$c \in G \cap H = \phi$, which is a contradiction and therefore X is connected.

Corollary # A non-subset X of R is connected iff X is either a point or an interval. In particular R is connected.

Proof Points and intervals in R are connected.
Now suppose that X is a connected subset of R. Let $a = glb\, X$ and $b = lub\, X$ (We let $a = -\infty$ if X is not bounded below and let $b = \infty$ if X is not bounded above). If $a = b$, then $X = \{a\}$ and this is a point. Suppose $a < b$. Let $c \in (a, b)$. There exist $x \neq y$ in X such that

$$a < x < c < y < b$$

If $c \notin X$, then as in above theorem X is disconnected. So $c \in X$
Thus $(a, b) \subseteq X$. If x $x < a$ or if $x > b$, then $x \notin X$. Therefore X is one of (a, b), $[a, b)$, $(a, b]$, or $[a, b]$

Connectedness & Continuity

Theorem # A top-space X is connected iff there does not exist a surjective continuous function f from X onto two point discrete space.

Proof # Suppose that X is connected. Also suppose that there is a continuous function $f : X \to \{a, b\}$, where $\{a, b\}$ is a two point

discrete space. Then by definition of continuity
$f^{-1}(\{a\}) = A$, $f^{-1}(\{b\}) = B$ are open in X.
Also

$$A \cup B = f^{-1}(\{a\}) \cup f^{-1}(\{b\})$$
$$= f^{-1}(\{a, b\})$$
$$= X$$

and $A \cap B = \phi$

This shows that X is disconnected which is a contradiction. Therefore no continuous surjective function from X to two point discrete space exists.

Conversely suppose that there does not exist a continuous surjective function from X onto the two point discrete space $\{a, b\}$. Let X be disconnected and $\{A, B\}$ be a disconnection of X. Then the mapping $f : X \to \{a, b\}$ defined by

$$f(A) = a \qquad f(B) = b$$

obviously $f^{-1}(\{a\}) = A$: $f^{-1}(\{b\}) = B$ which are open and hence f is continuous. Also f is onto. Thus f is a continuous surjective mapping from X to a two point discrete space $\{a, b\}$, a contradiction. Hence X is connected.

Characteristic function

Let (X, T) be a top-space and A be a non-empty, proper subset of X. A real valued function f on X, defined by

$$f(x) = 1 \qquad \forall x \in A$$
$$= 0 \qquad \forall x \in X \times A = A'$$

is called the characteristic function of A in (X, T)

Note: Characteristic function χ_A of A in (X,T) is continuous on X iff A is both open and closed.

Theorem # A subset A of a topological space (X,T) is connected iff there exists no continuous mapping $f: A \to R$ such that $f(A)$ consists of exactly two points.

Proof #. Let A be connected. If possible let there be a continuous mapping $f: A \to R$ such that $f(A) = \{a, b\}$; where $a, b \in R$, then $f^{-1}(\{a\})$ and $f^{-1}(\{b\})$ are disjoint closed sets in A. Further

$$A = f^{-1}(\{a\}) \cup f^{-1}(\{b\})$$

$\Rightarrow$ A is disconnected — a contradiction. Hence no such function is possible.

Conversely suppose that no continuous function $f: A \to R$ exists such that $f(A) = \{a, b\}$. We are to prove that A is connected.

Let A be disconnected and

$$A = A_1 \cup A_2 \text{, where } A_1 \text{ \& } A_2 \text{ are}$$

disjoint closed sets in A.

Then characteristic function χ_{A_1} defined by

$$\chi_{A_1}(x) = \begin{cases} 1 & \text{if } x \in A_1 \\ 0 & \text{if } x \in A_1^c = A_2 \end{cases}$$

~~Then $\chi^{-1}_{A_1}(\{1\}) = A_1$ & $\chi^{-1}_{A_1}(\{0\}) = A_2$~~

~~$\Rightarrow$ Inverse images of closed sets are closed~~

$\Rightarrow$ χ_{A_1} is continuous function on A which can be proved as

If k is any closed set in R containing 0 but not 1. Then

$$\chi^{-1}_{A_1}(k) = A_2 \text{, } A_2 \text{ is closed}$$

If k contains 1 but not 0, then

$$\chi^{-1}_{A_1}(k) = A_1 \text{, } A_1 \text{ is closed}$$

If contains none of 0 and 1 then

$$\chi_{A_1}^{-1}(K) = \phi \quad \phi \text{ is closed in } A$$

If k contains both 0, 1, then

$$\chi_{A_1}^{-1}(K) = A \text{, which is closed } A$$

and thus χ_{A_1} is continuous on A which is a contradiction. Hence A is connected (proved)

Comments # It is a trivial observation that any two discrete spaces with the same number of points are identical; for any one-to-one mapping of one onto other (there is at least one) is a homeomorphism and we may think of them as differing only in symbols used to designate their points. In this sense there is only one discrete space with any given number of points. The discrete two points space is obviously disconnected. In the theorem previous to above theorem we may take any two points discrete space usually it is taken as $\{0,1\}$

Theorem # A top-space (X, T) is connected iff every continuous function from X to a two point discrete space is constant.

Proof # Often two denotes discrete space with exactly two points. All such spaces are homeomorphic. For definiteness we take

$$(\{0,1\}, T') \text{, where } T' \text{ is discrete}$$

topology on $\{0,1\}$

Let X be connected and $f: X \rightarrow \{0,1\}$ be continuous. Let $c \in f(X) \subset \{0, 1\}$. Then $\{c\}$ being a singleton in discrete space $(\{0,1\}, T')$ is non-empty both open and closed. Since f is continuous, $f^{-1}(c)$ is non-empty

, both open & closed, subset of X. But X being connected $f^{-1}(C)$ cannot be a proper subset of X [X connected iff X itself is both open & closed].
Hence $f^{-1}(C) = X \Rightarrow f(X) = C$. This shows that f is constant.

Conversely let every conti function from X to {0, 1} be constant. Let $f: X \to \{0, 1\}$ be a continuous constant function and suppose that (X, T) be disconnected. Then there exist a non-empty proper subset A of X which is both open and closed. The set A' will also be both open and closed.

Now consider the characteristic function $\psi_A : X \to \{0, 1\}$ defined by

$$\psi_A(x) = 1 \quad \text{if } x \in A$$
$$= 0 \quad \text{if } x \in A'$$

Then $\psi_A^{-1}(\phi) = \phi$, $\psi_A^{-1}\{1\} = A$, $\psi_A^{-1}\{0\} = A'$, $\psi_A^{-1}\{0, 1\} = X$.

Thus the inverse image under ψ_A of every open subset of {0, 1} is an open subset of X. Consequently ψ_A is continuous but not constant.

This contradicts the hypothesis. We conclude that (X, T) is connected.

Theorem # Any continuous image of a connected space is connected i.e if (X, T) is connected and $f: (X, T) \to (Y, T')$ is continuous, then $(f(X), T'_{f(X)})$ is connected.

Proof # Let (X, T) be a connected space and (Y, T') be a top. space. Let $f: X \to Y$ be a continuous function. We are to prove that $(f(X), T'_{f(X)})$ is connected.

If f(X) is ... connected ...

exists two non-empty disjoint open subsets G, H of Y s.t

$f(X) \cap G \neq \phi$ $\quad$ $f(X) \cap H \neq \phi$

$f(X) \subseteq G \cup H$

Since G & H are non-empty, $f^{-1}(G) \neq \phi$, $f^{-1}(H) \neq \phi$ and because f is continuous, $f^{-1}(G), f^{-1}(H)$ are open subsets of X.

We know that

$$f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$$

$$\Rightarrow f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$$

and $f^{-1}(G) \cup f^{-1}(H) = f^{-1}(G \cup H) = f^{-1}(f(X)) = X$

$\because f^{-1}(G \cup H) \subseteq X$
But $f(X) \subseteq G \cup H$
$\Rightarrow X \subseteq f^{-1}(G \cup H)$
$\Rightarrow f^{-1}(G \cup H) = X$

$\Rightarrow \{ f^{-1}(G), f^{-1}(H) \}$ is a disconnection of X, which is a contradiction. Hence $f(X)$ is connected.

<u>OR</u>

If $(f(X)\ T'_{f(X)})$ is not connected, then there exists a continuous function (by a previous theorem which states that A subset 'A' of a top-space X is connected if there exists no continuous mapping $f: A \to R$ s.t. $f(A)$ is two pointic set) $g: f(X) \to R$ such that $g(f(X))$ is a two pointic set in R. But $g \circ f$ is continuous on X having two pointic image in R which shows that X is disconnected. This is a contradiction. Hence $f(X)$ must be connected.

<u>Corollary#</u> If $f: X \to Y$ is a homeomorphic mapping between (X, T) and (Y, T') and further

(X, T) is connected, then (Y, T') is connected. Thus connectedness is a topological property.

Theorem # The range of a continuous real function defined on a connected space is an interval.

Any subset of R is connected iff it is interval. $f(X) \subseteq R$

Proof # Let $f : X \to R$ be a continuous real function defined on a connected space X to the real line.
Since continuous image of a connected space is connected, $f(X)$ is a connected subset of R. Again a subset of real line is connected iff it is an interval. This implies that $f(X)$ being a connected subset of real line is connected. Hence proved)

Theorem # Let (X, T), (Y, T') be topological spaces and $f : X \to Y$ be continuous. If $C \subseteq X$ is connected, $f(C)$ is a connected subset of Y.   C is subset of X.

C connected C ⇒ f(C) is connected

Proof # Let $f(C)$ be disconnected. Then $\exists$ two non-empty disjoint open sets (in) G, H in Y such that

$$f(C) \cap H \neq \phi \qquad f(X) \cap H \neq \phi$$

and $f(C) \subseteq G \cup H$.

$\because$ f is continuous, G, H are non-empty open subsets of Y

$\therefore$ $f^{-1}(G)$, $f^{-1}(H)$ are non-empty and open

$$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$$

$$f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H) \subseteq X$$

$$f(C) \subseteq G \cup H \Rightarrow C \subseteq f^{-1}(G) \cup f^{-1}(H)$$

$$f(C) \cap H \neq \phi \qquad f(C) \cap H \neq \phi$$

$$\Rightarrow f^{-1}\{f(C) \cap H\} \neq f^{-1}(\phi) \neq \phi \qquad f^{-1}\{f(C) \cap H\} \neq \phi$$

$\Rightarrow C \cap f^{-1}(H) \neq \phi \qquad C \cap f^{-1}(G) \neq \phi$

$\Rightarrow \{f^{-1}(H), f^{-1}(G)\}$ is a disconnection of C which is a Contradiction. Hence $f(C)$ must be Connected.

OR

Let (X_0, T_0) be a discrete space. Let $g: f(C) \to X_0$ be Continuous. Then $g \circ f$ is a Continuous function from C to X_0. Since C is Connected, $g \circ f$ is Constant. [A top-space X is Connected iff every continuous function from X to discrete space is Constant). Therefore g must be Constant. Hence $f(C)$ is Connected.

Cor $X \times Y$ is connected $\Leftrightarrow$ X & Y are both Connected.

Proof # (i) Let $X \times Y$ be Connected. Since $Y \neq \phi$, the projection mapping $\phi: (x, y) \to x$ from $X \times Y$ to X is surjective. ϕ is also Continuous. Hence $\phi(X) = X$ is Connected. Similarly Y is Connected.

(ii) Conversely let both X & Y be Connected and (X_0, T_0) be a discrete space. Let $\phi: X \times Y \to X_0$ be Continuous. For each fixed $y \in Y$, $x \mapsto \phi(x, y)$ is Continuous function from X to X_0. Hence $\phi(x, y)$ is independent of x. Similarly ϕ is independent of y. So ϕ is Constant. Thus $X \times Y$ is Connected.

Theorem # Prove that a top-space (X, T) is disconnected iff there exists a Continuous map of X onto the discrete two-point space $\{0, 1\}$

Proof # Let (X, T) be disconnected. Then $\exists$ non-empty disjoint open sets G & H such that

$X = G \cup H \qquad G \cap H = \phi$

Consider a map $f: X \to \{0\ 1\}$ defined by

$$f(x) = 0 \quad \text{if } x \in G$$
$$= 1 \quad \text{if } x \in H$$

Then f is clearly onto

The only open subsets of $\{0,1\}$ are $\{0\}$, $\{0,1\}$, ϕ, $\{1\}$.

$$f^{-1}(0) = G \quad f^{-1}(1) = H \quad f^{-1}\{0,1\} = X, \quad f^{-1}(\phi) = \phi$$

Since G, H, X, ϕ are all open sets, it follows that f is continuous

So there exists a continuous function from X onto two-point discrete space $\{0,1\}$

Converse # Let $f: X \to \{0.1\}$ be continuous (onto) (same-same) mapping and assume that $(X.T)$ is connected. Since continuous image of a connected top-space is connected. It follows that continuous image of X i.e $f(X) = \{0.1\}$ is connected. This is wrong because every discrete space with more than one point is disconnected. Consequently $(X.T)$ must be disconnected.

Theorem # Prove that a top- space is connected iff for every continuous function $f: X \to R$, $f(X)$ does not consist of two distinct real numbers.

Proof # Let f be a continuous function of X into R such that.

$$f(X) = \{p\} \cup \{q\}, \quad p \neq q$$

Choose two disjoint open sets in R as under.

$$J =]p-\epsilon \,.\, p+\epsilon[\quad K =]q-\epsilon \quad q+\epsilon[\quad \epsilon = \frac{|p-q|}{3}$$

Since f is continuous, $f^{-1}(J)$ and $f^{-1}(K)$ are open and that

$$f^{-1}(J) \cup f^{-1}(K) = f^{-1}(J \cup K)$$
$$= X \quad \Rightarrow X \text{ is disconnected.}$$

Hence $f(X)$ does not consists of two distinct real numbers.

(ii) Converse: Let $f(X)$ does not consists of two real distinct points and X is disconnected. with disconnection $\{G, H\}$ such that

$$X = G \cup H \qquad G \cap H = \phi$$

Let us define $f: X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in G \\ 1 & \text{if } x \in H \end{cases}$$

Then f is clearly continuous and $f(X) = \{0, 1\}$ i.e consists of two distinct real points which is a contradiction. Hence X must be connected.

Theorem # Let (X, T) & (Y, T') be two top-spaces, C a connected subset of X and a function $f: X \to Y$. Let f_c denotes the restriction of f to set C. If f_c is continuous, prove that $f(C)$ connected..

Proof # Assume to the contrary. Let $f(C)$ be disconnected and $D = f(C)$ and $\psi = f_c$

Since D is disconnected, $\exists$ a non-empty proper subset A of D which is both open and closed in D. $\because D \subseteq Y$, there exists a set G open in Y and set F closed in Y such that

$$A = G \cap D \qquad A = F \cap D$$

$$\psi^{-1}(A) = \psi^{-1}(G \cap D) = \psi^{-1}(G) \cap \psi^{-1}(D)$$
$$= \psi^{-1}(G) \cap C = \psi^{-1}(G)$$

Similarly $\psi^{-1}(A) = \psi^{-1}(F)$

Thus $\psi^{-1}(A) = \psi^{-1}(G) = \psi^{-1}(F)$.

Since $\psi = f_c$ is continuous, $\psi^{-1}(G)$ is open and $\psi^{-1}(F)$ is closed.

It follows that $\psi^{-1}(A)$ is a non-empty proper subset of C which is both open and closed

in C. But this means that C is disconnected, which contradicts the hypotheses. Hence $D = f(C)$ is connected.

Theorem # Let (X, T) be a topological space. Then following are equivalent.

(a) X is connected.

(b) X is not expressible as union two non-empty disjoint closed sets.

(c) The only subsets of X which are both open and closed are X & ϕ

(d) Every proper subset of X has non-empty boundary

(e) There is no continuous surjective function from X to a two point discrete space $\{0, 1\}$

Proof # (a) $\Rightarrow$ (b)

$-X = A \cup B$
$A \cap B = \phi$

Let X be connected and X be expressed as union of two non-empty disjoint closed sets F_1 & F_2. Then clearly X is disconnected which is contrary to (a). Hence X cannot written as union of two disjoint non-empty closed sets. Thus (a) $\Rightarrow$ (b)

(b) $\Rightarrow$ (c)

Suppose that X cannot be written as union of two non-empty disjoint closed sets. We are to prove that the only subsets of X which are both open and closed are X & ϕ

Let A be a proper subset of X which is both open and closed. Then A' will also be both open and closed and

A is different from X & ϕ

$$X = A \cup A' \qquad A \cap A' = \phi \text{ (disjoint)}$$

$\Rightarrow$ X is written as union two non-empty disjoint closed subsets of X because $A \neq \phi, X$ and therefore $A' \neq \phi$

X has no proper subset which is both open & closed. X, ϕ are only open, closed

This is contrary to (b). Hence X & ϕ are only sets which are both open and closed.

(c) $\Rightarrow$ (d)

Suppose that the only subsets of X which are both open and closed are X & ϕ. We are to prove that every proper subset of X has non-empty boundry.

Suppose on the contrary that non-empty proper subset Subset A (i.e $A \neq X$, $A \neq \phi$) of X has an empty boundry i.e

$$A^b = \phi$$

Then A is open and

$$A' = X - A \quad \text{is closed.}$$

Now $A^b = \phi$

$$\Rightarrow \bar{A} \cap \bar{A^c} = \phi$$

Let p be a limit point of A, then $p \in \bar{A}$. But $p \notin \bar{A^c}$ $(\because \bar{A} \cap \bar{A^c} = \phi)$

In particular $p \notin A^c$ because p is not limit point A^c

So $p \in A$

$\Rightarrow$ A contains all of its limit points. Hence A is also closed.

Thus A is a proper subset of X which is both open and closed. This is contrary to (c). Hence the only subsets of X which are both open and closed are X & ϕ.

(d) $\Rightarrow$ (e)

Suppose that every proper subset of X has a non empty boundry. We are to prove that there is no continuous surjective mapping from X to a two points discrete space $\{0, 1\}$.

On the contrary let $f: X \to \{0\ 1\}$ be a continuous surjective function. Then $\because$ Singleton in discrete space are both open and closed, therefore $\{0\}$ is both open and closed. Since f is continuous, therefore $f^{-1}(\{0\})$ is also both

open and closed in X. Hence $(f^{-1}\{0\})^b = \phi$ [If $A \subseteq X$, then $A^b = \phi$ iff A is both open and closed] which is Contrary to (d). Hence there is no continuous surjective function from X to a two point discrete space $\{0,1\}$

(e) $\Rightarrow$ (a)

Let there be no continuous surjective function from X to two point discrete space $\{0,1\}$. We are to prove that X is Connected. Let X be disconnected. There $\exists$ two non-empty disjoint open sets G, H Such that

$$X = G \cup H$$

Define a function $\psi : X \to \{0, 1\}$ as

$$\psi(x) = 0 \quad \text{if } x \in G$$
$$= 1 \quad \text{if } x \in H = G'$$

Then ψ is clearly surjective and continuous function from X to two point discrete space $\{0,1\}$ which is a Contradiction to (e). Hence X is Connected. (Proved)

Theorem # Let (X T) be a top- space. The following are equivalent.

(a) # (X.T) is disconnected

(b) # There exist two non-empty disjoint subsets A and B, both open [or both closed] in (X T) Such that $X = A \cup B$

(c) There exists a proper Subset of X which is Simultaneously open and closed in (X T)

Proof# (X.T) is disconnected iff for some $A \neq \phi$ $B \neq \phi$, we have.

$$A \cap \bar{B} = \phi \quad , \quad \bar{A} \cap B = \phi \quad X = A \cup B$$

Assume that (a) is true. Then $\exists$ two non-empty sets A & B Such that

$A \cap \bar{B} = \phi$ $\bar{A} \cap B = \phi$ $X = A \cup B$

from $A \cap \bar{B} = \phi$ it follows that

$A = X - \bar{B} = (\bar{B})'$

$\Rightarrow$ A is open in $(X\ T)$

from $\bar{A} \cap B = \phi$ it follows that

$B = X - \bar{A} = (\bar{A})'$

$\Rightarrow$ B is open in $(X\ T)$

Since $A \cap B \subseteq A \cap \bar{B} = \phi$, therefore

$A \cap B = \phi$

Also $X = A \cup B$

Thus $(a) \Rightarrow (b)$

Trivially open sets can be replaced by closed sets. If A is open, then $B = X - A$ is closed; If B is open $A = X - B$ is closed etc.

$(b) \Rightarrow (c)$

Assume (b) is true and let A & B be both closed (or both open). Since $A = X - B$ and B is closed, so $A = B'$ is open. Thus A is open and simultaneously closed in (X, d).

So $(b) \Rightarrow (c)$.

$(c) \Rightarrow (a)$

Assume (c) is true and let A be a proper subset of X and that A is simultaneously open and closed in X. Let $B = X - A$, then $X = A \cup B$ & $A \cap B = \phi$ so $\{A\ B\}$ is a disconnection of X. Thus $(c) \Rightarrow (a)$.

Connectedness Between Points

Any two points a, b in a top-space X are said to be connected if they both belong to same connected subset of X.

Symbolically $a \simeq b$ (a is connected with b)

Theorem # The relation of connectedness between points of a top-space X is an equivalence relation.

Proof # (i) Reflexive #

Relation is reflexive because for any $a \in X$, $a, a \in \{a\}$, where $\{a\}$ is a connected subspace of X i.e each element is connected with itself.

(ii) Symmetric #

For any $a, b \in X$, $a \simeq b \Rightarrow \exists$ a a connected subset C of X which contains a and b.

Hence $b \simeq a$ because C also contains b & a.

(iii) Transitive #

Relation is transitive because $a \simeq b$, $b \simeq c \Rightarrow \exists$ connected subsets C_1 & C_2 such that

$a, b \in C_1$ $\quad b, c \in C_2$

Now

$a, c \in C_1 \cup C_2$

$\because C_1 \cap C_2 \neq \emptyset$

$\therefore C_1 \cup C_2$ is also connected.

$\Rightarrow a \simeq c$.

Thus connectedness between points is an equivalence relation.

Comments # Because any equivalence relation partititions the set into equivalence classes. The equivalence classes made by the relation of $\simeq$ are the maximal connected subsets and are known as components.

Components In a Top Space

Introduction # When a top- space is not connected it is natural that we should attempt to obtain some information about the various " pieces " into which it can be separated. The maximal connected subsets of the space are particularly intervresting.

Component # A maximal connected subspace (subset) of a top- space i.e a connected sub-space which is not properly contained in any larger connected subspace, is called a component of the space

Examples # (a) A connected space clearly has only one compoent namely space itself.

(b) In a discrete space each singleton (or each point) is a compoent.

Comments # We have proved that the relation of connectedness of elements of a top- space is and equivalence reation. Also each equivalence relation in a space divides the space into mutually disjoint equivalent classes, For any $a \in X$, let C_a be the class of all elements of X which are connected with (a), then C_a is maximal connected subset of X containing (a) and so is a component of. As such all equivalence classes introduced by the relation of connectedness of elements are components of X.

Component at a Point # If $x \in X$, a component

at x is always that component in the space X that contains the point x.

Quasi-Component At a Point

A quasi-Component at a point x in X is the intersection of all clopen (i.e both open and closed) sets in X containg the given point.

Component of a Subset

Let E be a subset of a top-space X. If $C(E,x)$ or C_x^E denotes be the union of all connected subsets of E containg x, then $C(E,x)$ is connected [$\cup C_\alpha$ is connected if $\cap C_\alpha \neq \phi$] and is the largest connected subset of E containing x. Hence $C(E,x)$ is called componet of E corresponding to x.

Theorem # The componets of a top-space (X, T) are closed subsets of X. Any two components of X are either identical or disjoint.

Proof # Let C be any component of of X. Since C is a component, it is a maximal connected set in X. Now $\bar{C}$ is connected (closure of a connected subset is connected)

$\because$ C is maximal connected

$\therefore$ $\bar{C} \subseteq C$

But by definition

$C \subseteq \bar{C}$

$\Rightarrow$ $C = \bar{C}$ Hence C is closed.

Next let A & B be two Components of X. If $A \cap B \neq \phi$, then $A \cup B$ is connected.

Now $A \subseteq A \cup B$ $B \subseteq A \cup B$

But A, B are maximal connected subsets of X. Hence a Contradiction. Hence Componets are either identical or disjoint.

Comments # (1) This theorem may be rephrased to subsets of a top-space. Thus, the compnents of any subsets E of X are closed in E

(2) We know that if (Y, T_Y) is a subspace of $(X.T)$, Then

$T_Y \subseteq T$ iff $Y \in T$

So this Theorem can be rephrased for closed sets i-e Components of closed sets are closed in X.

(3) The Components of an open set are not open in general.

Let $(Q\ T_Q)$ as subspace of (R, T), where T_Q is induced topology from real numbers. Then each point in Q is a Component of the entire space (an open set, certainly) but the individual points are not open in this topology. This example also shows that a Component and its Complement may not form a separation of a space.

Theorem # In a top-space (X, T)

(a) # The relation of connectedness between any two points of X is an equivalence relation

(b) # The disjoint equivalence classes constitute all the Components of the space X

(c) # The Components of X form a partition of X

Proof # (a) Already proved.

(b) # Let C be an equivalence class. Choose $x_0 \in C$. For every $x \in C$, we know that $x \simeq x_0$ and so there exists a connected set A_x containing x & x_0.

If possible, let $A_x \cap D \neq \phi$ where D is an equivalence class other than C. but

$$A_x \cap C \neq \phi \text{ as well}$$

Let $x_1 \in A_x \cap D$ and $x_2 \in A_x \cap C$.

Then x_1, x_2 belonging to the same connected set A_x are connected i.e $x_1 \simeq x_2$. But this leads to a contradiction because x_1 & x_2 belong to two different equivalence classes D and C.

Hence $A_x \cap D = \phi$ for any equivalence class D other than C. Thus $A_x \subseteq C$. obviously

$$C = \bigcup_{x \in C} A_x.$$

$\because$ $x_0 \in \cap\{A_x \mid x \in C\}$

$\therefore$ $C = \bigcup_{x \in C} A_x$ is connected.

C is also maximal connected. In fact if D is another connected set containing C, then $D \cap C \neq \phi$ and thus by argument above, D can not intersect any other equivalence class and hence $D = C$. Thus every equivalence class is a component.

That every component is an equivalence (for $\simeq$) is obvious from definition

(c) # Let $x \in X$ and let C_x be the union of all connected sets which contain x. Then C_x is is a component of X. (by definition)

Since the family $\{C_x : x \in X\}$ consists of all components of X, we must have.

$$X = \cup\{C_x : x \in X\} \longrightarrow (1)$$

The members family form a partition of X.

Let C_p and C_q be any two components of X w.r.t two different points $p, q \in X$ $(q \neq p)$.

If $C_p \cap C_q = \phi$, the job is done. If $C_p \cap C_q \neq \phi$, then let $y \in C_p \cap C_q$

$$\Rightarrow y \in C_p \ \& \ y \in C_q$$

Now C_p, C_q are connected sets each containg y, and C_y is a component containing y, So

$$C_p \subseteq C_y \quad \& \quad C_q \subseteq C_y$$

But C_p, C_q being components are maximal connected sets, they canot be contained in larger connected subsets of X. This means

$$C_p = C_q = C_y$$

Consequently, any two components of X are either disjoint or identical. So (1) form a partition of X (proved)

Theorem # Let X be a top-space. Then.

(a) # each element of X is contained in an exactly one component of X

(b) # Each connected subset of X is contained in a component of X

(c) # A connected subset of X which is both open and closed is a component

(d) # Components need not be open sets

Proof # First we prove that components are disjoint. Let E_1 & E_2 be two distinct components and.

$$E_1 \cap E_2 \neq \phi$$

Then $E = E_1 \cup E_2$ is connected and $E_1 \subseteq E$, $E_2 \subseteq E$ i.e E is connected superset of E_1, E_2 which is not possible because E_1 & E_2 are maximal connected sets. Hence

$$E_1 \cap E_2 = \phi$$

(a) # Let $x \in X$, then singleton $\{x\}$ is non-empty connected subset of X [$\{x\}$ is connected because it can not divided into non-empty disjoint parts]. We can extend $\{x\}$ to a component E_x of X. E_x is unique because components are disjoint. Thus the element x belongs to component E_x and to no other component.

<u>OR</u>

Let $x \in X$ and $\{C_\alpha : \alpha \in I\}$ be the class of all connected subsets of X containing x. Put

$$C = \bigcup_{\alpha \in I} C_\alpha$$

Then since $x \in C_\alpha$, $\forall \alpha \in I$, therefore

$$\bigcap_{\alpha \in I} C_\alpha \neq \phi$$

So $C = \bigcup_{\alpha \in I} C_\alpha$ is connected and contains x

We prove that C is a component. Suppose D is a connected subset such that

$$C \subseteq D$$

Then $x \in D$, D is one of C_α,s. Thus

$$D \subseteq C$$

This means that $D = C$. i.e C is not contained in any other connected subset. Hence C is a component. So every element of X belongs to a component.

Also if $x \in C_1$, $x \in C_2$ where C_1, C_2 are Components of X. Then

$$x \in C_1 \cup C_2$$

Now $C_1 \cup C_2$ is Connected because $C_1 \cap C_2 \neq \phi$
$\Rightarrow$ C_1, C_2 are Subsets of another Connected subset $C_1 \cup C_2$ which is Contrary to the fact that C_1 & C_2 are Components. Hence every element of X is contained in exactly one Component of X.

(b) # Let A be a Connected subset of X. Consider the family $\{E_\alpha\}_{\alpha \in I}$ of all connected Super sets of A.

$\{E_\alpha\}$ family of subsets of A.

This family is non-empty because $A \subseteq A$ and so

$$A \in \{E_\alpha\}_{\alpha \in I}$$

Let $E = \bigcup E_\alpha$

Since $\bigcap_{\alpha \in I} E_\alpha = A$

Therefore, E is Connected Subset. E is Component
If $E \subseteq B$ & B is a (Component) Connected, Then
$A \subseteq E \subseteq B$ and so $B \in \{E_\alpha\}$. Thus $B \subseteq E$
and we have $E = B$, i.e.

maximal and become component $\leftarrow$ E is not Contained properly in any other Connected Subset. Thus A is Contained in a Component.

(C) # A Connected Subset which is clopen is a Component of X

clopen: open & closed

Proof # Suppose that A is a Connected Subset of X which is both open and closed. Then by (b) A is contained in a Component C of X i.e.

$$A \subseteq C.$$

We prove that $A = C$ i.e A is an improper subset of C.

If $A \neq C$ i.e if A is a proper Subset of C

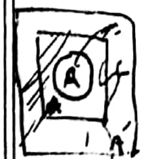
, then

$$C = (C \cap A) \cup (C \cap A') \text{ i.e}$$

$$C \subseteq A \cup A' \qquad A \cap A' = \phi \quad \text{disjoint}$$

A, A' are both open

$\Rightarrow C$ is disconnected which is a Contradiction.



Hence $C = A$

Thus A is a Component of X.

(d) # Components need not be open sets.

Consider the top space Q of rationals. → $Q \subseteq R$

Let $D \subsetneq Q$ Consists of at least two points say p & q, with $p < q$. Then there always exists a real number $x \in R - Q$ such that [illegible] irrational No.

$$p < x < q$$

Let $G =]-\infty \; x[$ and $H =]x \; +\infty[$

$\therefore D_1 = G \cap D \qquad D_2 = H \cap D$

Then $D_1 \neq \phi \qquad D_2 \neq \phi$ → [illegible]

$D_1 \cap D_2 = D \cap (G \cap H) = \phi \;, D = D_1 \cup D_2$

, D_1, D_2 are open in D.

$\Rightarrow D$ is disconnected

It follows that every subset of Q which contains at least two points is disconnected.

Since every singleton is always connected, so componets of Q are the singleton sets. No singleton in Q is open in Q. because if $z \in Q$ and G is any open subset of R, then $\{z\} \neq G \cap Q$, since every open subsets of R contains infinitely many points of R

We conclude that componets of Q are singletons, which are not open in Q.

[illegible] more than one [illegible] Subset [illegible] Q

[illegible] contain [illegible] disconnected [illegible]

Theorem # Let X be a top-space, x a point of X and B^x the family all connected subsets of X containing x. Then

$$K^x = \cup\{P : P \in B^x\}$$

is a component of X

Proof # Since $\{x\}$ is connected and $x \in \{x\}$, $\{x\} \in B^x$ and $B^x \neq \phi$.

obviously $x \in \cap\{P : P \in B^x\}$ i.e $\cap\{P : P \in B^x\}$ is non-empty. Hence $K^x = \cup\{P : P \in B^x\}$ is connected. If $K^x \subset Q$ and Q is connected, then $x \in Q$ and hence $Q \in B^x$. Consequently $Q \subset K^x$. Thus K^x is a maximal connected set in X. Hence K^x is a component of X

Theorem # A space is connected iff any two points are connected. OR A space is connected iff it has only one component.

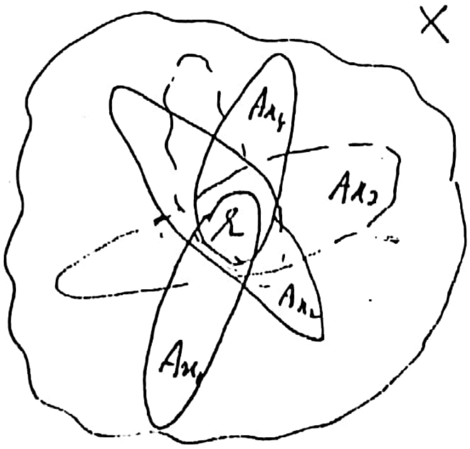
Proof # Let $(X\ T)$ be a connected space and $a, b \in X$ be any two points. Then clearly X is a connected set containg a & b and hen $a \overset{c}{\sim} b$

Conversely Let any two points of X be connected with each other. Choose $\ell \in X$

For any $x \in X$,

ℓ, x are connected and $\exists$ a connected subset A_x such that

$\ell, x \in A_x$



$\Rightarrow X = \cup\{x : x \in X\}$

$\subseteq \bigcup_{x \in X} A_x \subseteq X$

$\Rightarrow X = \bigcup_{x \in X} A_x$

Now $\varepsilon \in A_x \quad \forall x \in X$

$\Rightarrow \bigcap_{x \in X} A_x \neq \phi$

Hence $X = \bigcup_{x \in X} A_x$ is Connected

Next we prove that X is Connected iff it has only one Component.

Let X be Connected. Then clearly X is a Component. We prove That this is the only one component. Let C be any other Component. Then C is a maximal Connected Subset. So

$$X \subseteq C$$

But $C \subseteq X$

$$\Rightarrow X = C$$

Thus there is only one component

Conversely let X has only one Component we prove X is connected. If X is not Connected, then X may have more Than one Components. Hence X is Connected.

Totally Disconnected Space

(1) A top-space X is totally disconnected iff every pair of distinct points of X can be separated by a disconnection of X i.e for any two distinct points x, y of X, there is a disconnection $\{A, B\}$ of X Such that $x \in A$ and $y \in B$

(2) A top-space is said to be totally disconnected iff all non-empty, non-singleton subsets are disconnected.

3) A top-space X is called totally disconnected if every quasi-component of this space contains only one point.

<u>Example</u> # Every discrete space is totally disconnected.

Let (X, T) be any discrete space and x, y be any two distinct points in X. Then A = {x}, B = X − {x} are two non-empty disjoint open sets with

$$X = A \cup B$$

such that

$$x \in A, \quad y \in B$$

Thus a disconnection of X, viz {A, B} exists such that $x \in A$, $y \in B$. Consequently (X, T) is totally disconnected.

<u>Example</u> # The set $\mathbb{Q}$ (rational numbers) and $R \setminus \mathbb{Q} = \mathbb{Q}'$ (The set of irrational numbers) are totally disconnected spaces with induced topologies

<u>Sol</u> # (a) Let $x, y \in \mathbb{Q}$, $x \neq y$, $x < y$. Then ∃ a point $z \in \mathbb{Q}'$ such that

$$x < z < y$$

$$A =]-\infty, z[\cap \mathbb{Q} \qquad B =]z, +\infty[\cap \mathbb{Q}.$$

$$[A \neq \phi \quad B \neq \phi, \quad A \cap B = \phi]$$

Sets A, B are open in $\mathbb{Q}$ and

$$\mathbb{Q} = A \cup B \qquad x \in A \quad y \in B$$

Consequently $\mathbb{Q}$ is totally disconnected.

(b) Let $x, y \in \mathbb{Q}'$, $x \neq y$, take $x < y$. Then ∃ a point $z \in \mathbb{Q}$ such that

$$x < z < y$$

take $A =]-\infty, z[\cap \mathbb{Q}'$. $B =]z, +\infty[\cap \mathbb{Q}'$

obviously, A & B are non-empty, disjoint open sets in Q' such that

$$Q' = A \cup B \quad x \in A, \; y \in B$$

It follows that Q' is totally disconnected.

Example # The cantor set is another totally disconnected subspace of real line.

Theorem # A totally disconnected space is a Hausdorff space.

Proof # Let X be a totally disconnected space. Then for each pair of distinct points in X, there exists a disconnection in X.

Let $x, y \in X \quad x \neq y$

Let $\{A, B\}$ be a disconnection of X such that

$x \in A \quad y \in B$.

As A & B are open sets and $A \cap B = \phi$

Thus every pair of distinct points in X is separated two disjoint open sets. Hence X is Hausdorff space

Theorem # Prove that the components of totally disconnected top-space are singleton sets.

Proof # To obtain the required result it is sufficient to show that every subset of X containing more than one point is always disconnected.

Let (X, T) be a totally disconnected space and Y be a subspace containing more than one point. Let x, y be distinct points in Y and let $\{A, B\}$ be a disconnection of X such that

$x \in A \quad y \in B$

Let $G = Y \cap A$ $\quad H = Y \cap B$

Now $G \neq \phi$ as $x \in G$ and $H \neq \phi$ as $y \in H$

$G \cap H = (Y \cap A) \cap (Y \cap B)$

$= Y \cap (A \cap B) = Y \cap \phi = \phi$

$G \cup H = (Y \cap A) \cup (Y \cap B)$

$= Y \cap (A \cup B)$

$= Y \cap X = Y$

It follows that $\{G, H\}$ is a disconnection of Y. Hence Y is disconnected.

It means that only connected subsets are singletons.

Theorem # Let X be a Hausdorff space. If X has an open base whose sets are also closed, then X is totally disconnected.

Proof # Let x & y be two distinct points in X. Since X is Hausdorff, $\exists$ an open set G such that

$x \in G$ and $y \notin G$.

Let β be a base whose sets are also closed.

$\because$ G is an open set

$\therefore$ There exists an basic open set B such that

$x \in B \subseteq G$

By hypothesis B is open as well as closed

Therefore B' is also open as well as closed.

$x \in B$ $\quad x \notin B'$ and $y \in B'$

Also $X = B \cup B'$

Thus $\{B, B'\}$ is a disconnection of X which (contains) separates x & y

Hence X is totally disconnected.

Note # If the space X in this theorem is also compact, then the implication can be reversed, and the two conditions are equivalent

Theorem # Let X be a Compact Hausdorff space. Then X is totally disconnected $\Leftrightarrow$ it has an open base whose sets are also closed.

Proof # Let X be a Hausdorff space, then by previous theorem X has an open base whose sets are also closed.

Conversely let X be totally disconnected. We prove that class of all subsets of X which are both open and closed forms an open base.

Let x be a point and G be an open set containing x. We are to produce a set B which is both open and closed such that

$$x \in B \subseteq G$$

If $G = X$, then $B = X$ satisfies the requirement. So

let $G \neq X$

G' is is a closed subspace of X. Since X is Compact, therefore G' is also compact because a closed subspace of a Compact space is Compact.

Since X is totally disconnected, for each point y in G', there exists a set H_y which is both open and closed and contains y but not x. G' is Compact, so there exists some finite class of H_y's which we denote by $\{H_1, H_2, H_3 \dots H_3\}$ such that

$$G' \subseteq \bigcup_{i=1}^{n} H_i \quad , \quad x \notin \bigcup_{i=1}^{n} H_i$$

$$\text{Let } H = \bigcup_{i=1}^{n} H_i$$

Since it is a finite union of sets which are both open and closed, H is also both open and closed.

$$\because \quad x \notin H$$

$$\therefore \quad x \in H' = B$$

$$\text{and} \quad B \subseteq G$$

$G' \subseteq H$
$\Rightarrow H' \subseteq G$

Thus for each point x of X, an open set G containing x there is a set B which is both open and closed such that

$$x \in B \subseteq G$$

$\Rightarrow X$ has a base whose sets are both open and closed.

Product of Connected Spaces

Theorem Let (X, T), (Y, T') be topological spaces, then
(i) if $(X \times Y, T \times T')$ is connected then so are X & Y
(ii) If X and Y are both connected, then so is $X \times Y$.

OR

$X \times Y$ is connected iff X and Y are both connected.

Proof # (i) Let $X \times Y$ be connected we prove that X and Y are both connected.
Let P_x, P_y denote, as usual, the projection maps.

$$\left.\begin{array}{l} P_x : X \times Y \longrightarrow X \\ (p, q) \longrightarrow p \end{array}\right\} \qquad \left.\begin{array}{l} P_y : X \times Y \longrightarrow Y \\ (p, q) \longrightarrow q \end{array}\right\}$$

We know that P_x, P_y are continuous and

$$P_x(X \times Y) = X \qquad P_y(X \times Y) = Y$$

Also continuous image of connected space is connected. Therefore X and Y are connected.

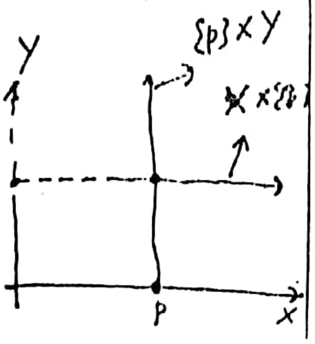
(ii) Converse of (i)

Let X and Y be connected. We prove that $X \times Y$ is connected.

Let $p \in X$ and let $A_p = \{p\} \times Y$

The mapping

$$\left.\begin{array}{l} h : Y \longrightarrow A_p \\ q \longrightarrow (p, q) \end{array}\right\}$$ is obviously



homomorphism. Hence, Y being connected so is A_p

Let $q_0 \in Y$ and $B_{q_0} = X \times \{q_0\}$ is similarly connected.

Now

$$A_p \cap B_{q_0} = (p, q_0) \neq \phi$$

So $B_{q_0} \cup A_p = H_p$ is connected for each $p \in X$

Take $H = \bigcup_{p \in X} H_p$

obviously $B_{q_0} \subset (\bigcap H_p)$ and H is connected

But

$$X \times Y = \bigcup_{p \in X} H_p$$

Hence $X \times Y$ is connected

Note # By induction on n, we can prove the above theorem (by) for finite connected spaces

Theorem # Let $\{X_\alpha : \alpha \in I\}$ be a collection of topological spaces. Then

(i) If $(\prod_{\alpha \in I} X_\alpha, T)$, where T is the product topology, is connected, then so is X_α for each α

(ii) If each X_α is connected, then so is $(\prod_{\alpha \in I} X_\alpha, T)$

Proof # Let $(\prod_{\alpha \in I} X_\alpha, T)$ be connected.

Consider projection mappings

$$P_\alpha : \prod_{\alpha \in I} X_\alpha \to X_\alpha$$

defined by

$$P_\alpha (x_1, x_2, \dots x_i \dots x_\alpha \dots) = x_\alpha \quad \forall \alpha \in I$$

Then projection mappings are continuous and

$$P_\alpha (\prod_{\alpha \in I} X_\alpha) = X_\alpha \quad \forall \alpha \in I$$

Hence each X_α is also connected because it is continuous image of a connected space.

(ii) Let $\hat{b} = (b_\alpha)$ be an arbitrary but fixed point of $\prod_{\alpha \in I} X_\alpha$. Let $F = \{\alpha_1, \alpha_2 \dots \alpha_n\} \subset I$

We called all points $x = (x_\alpha)_{\alpha \in I}$ from $\prod_{\alpha \in I} X_\alpha = X$ s.t. $x_\alpha = b_\alpha$ for $\alpha \neq \alpha_1, \alpha_2 \dots \alpha_n$ name the set as X_F.

We consider the mapping

$f: X_{\alpha_1} \times X_{\alpha_2} \times \dots X_{\alpha} \to X_F = (x_{\alpha_1}, x_{\alpha_2} \dots x_{\alpha_n}) \to (y_\alpha)_{\alpha \in I}$

where $y_\alpha = x_\alpha$ for $\alpha = \alpha_1, \dots \alpha_n$ and $y_\alpha = b_\alpha$ for other values of α. obviously f is a bijection, carries a basis element for $X_{\alpha_1} \times \dots X_{\alpha_n}$ to a basis element of X_F; f is continuous as well and is thus a homomorphism

Let $Y = \bigcup \{X_F : F$ is any finite subset of $I\}$ obviously $X_{\alpha_1} \times X_{\alpha_2} \times X_{\alpha_3} \times \dots X_{\alpha_n}$ is connected. and so is X_F and Y is connected since $b = (b_\alpha)$ is a member of each X_F.

We show that $\bar{Y} = X$ which would prove that X is connected.

Let $x = (x_\alpha)_{\alpha \in I} \in X$ and let $U = \prod U_\alpha$ be an arbitrary basic open set containing x. By definition $U_\alpha = X_\alpha$ except for a finite no of indices say $\alpha_1, \alpha_2 \dots \alpha_n$

Let $y_\alpha = \begin{cases} x_\alpha & \text{for } \alpha = \alpha_1, \alpha_2 \dots \alpha_n \\ b_\alpha & \text{for all } \alpha \in I \setminus \{\alpha_1, \alpha_2 \dots \alpha_n\} \end{cases}$

obviously $(y_\alpha) \in Y$ because it belongs to the space X_F where $F = \{\alpha_1, \alpha_2 \dots \alpha_n\}$; (y_α) is a point of Y as well. Since $y_\alpha = x_\alpha \in U_\alpha$ for $\alpha = \alpha_1, \dots \alpha_n$ and $y_\alpha = b_\alpha \in X_\alpha$ for all other values of α. Then $(y_\alpha) \in U \cap Y$ and thus $(x_\alpha) \in \bar{Y}$